MODERN SOLID GEOMETRY

WELLS AND HART

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MODERN SOLID GEOMETRY

GRADED COURSE

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PREFACE

THE predecessor of this Modern Solid Geometry, the Wells and Hart Solid Geometry (1916), anticipated by several years practically all the recommendations of the National Committee pertaining to this subject. This was due in part to the fact that the text incorporated the spirit of Hilbert's Foundations of Geometry; and, secondly, to the pedagogical ideals which actuated the author.

The Modern Solid Geometry is a refinement of the ideals of its predecessor, the logical outcome of ten more years experience of the author in teaching the subject.

The fundamental aim of the author is to offer a mathematically sound treatment of the subject in thoroughly modern pedagogical form. The first part of this ideal is responsible for the definitions and statements of the theorems in the text. Essentially these definitions are the ones which appeared in the Solid Geometry (1916), and which have been recommended since then by the National Committee.

Attention is directed to: the omission of a definition of Surface (§ 407); the definitions of Diedral Angle (§ 445) and Polyedral Angle (§ 469); of a Solid (§ 490); of Polyedron (§ 491); of Cylindrical Surface (§ 555); of Cylinder (§ 560); and of Sphere (§ 608). Attention is directed also to the clear-cut statement of the postulates of the plane (§ 408); to the frank acceptance of § 507 as a definition; to the introduction of Cavalieri's Theorem as an unproved theorem, and its use in § 533, § 576, and elsewhere in alternative proofs of certain theorems; to the informal treatment of the concept of "limit" and the use of it informally in § 531, § 566, § 571, etc.; to the more accurate statement and proof in § 562 and § 586 of two theorems usually given with less care.

The unique characteristic of the text is *its organization in terms of a minimum course supplemented by Optional Topics.* Adaptation of instruction to the capacities of students is the foremost need and ideal of current educational effort. This text provides the logical means of accomplishing this purpose.

The minimum course includes all the theorems indicated as Fundamental by the College Entrance Examination Board. These are marked by a star (*) in this text, as in their list. The other theorems of the minimum course all appear in the list recommended by the Board. Most of the propositions are accompanied by one or more carefully selected exercises. This minimum course is probably the briefest one, logically organized, which is available for this subject. To complete the full requirements of the College Entrance Examination Board, this minimum course should be supplemented by study of the following Optional Topics: Topic B, pp. 410–412; Topic A, pp. 444–450; and Topic D, p. 456.

To limit the opportunities of able pupils to this minimum course is *as grave a pedagogical error* as to overtax the less capable ones. This text provides for the able pupils by means of its *Optional Topics* and *its additional exercises.* Examples of such optional topics appear on pages 346–351; pages 378–389; etc. This subject matter either appears as subsidiary in the list of the College Entrance Examination Board or of the National Committee, or it is too difficult for less able pupils. All of it appears in some form in current texts. The *uniqueness of this text* consists in definitely setting this material apart as not required in the logical development of the minimum course.

The second means of providing for individual needs is the list of additional examples with *the unique system of cross reference to them.* See for example the footnote on page 321.

The chapter on Elementary Trigonometry is included for two reasons: classes preparing for *comprehensive* or ordinary examinations of the College Entrance Examination Board will find this material useful whether or not pupils have studied like material earlier in their course; other classes may wish to use it to supplement a brief course in solid geometry.

No effort has been spared to make the book attractive and durable. Important topics and almost all propositions start at the top of a page, and every demonstration is completed without turning to the next page. Authorities, notes, remarks, and exercises are printed in a second size type in order to make the main body of the text more emphatic.

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SYMBOLS AND ABBREVIATIONS

It is not necessary to learn any of these symbols or abbreviations until they are introduced in the text.

Symbols

$\frown$,	arc.
$=$,	is equal to; equals.
$>$,	is greater than.
$<$,	is less than.
$\parallel$,	is parallel to; parallel.
$\perp$,	is perpendicular to.
$\perp$,	perpendicular.
$\sim$,	is similar to.
$\cong$,	is congruent to.
$\angle$,	angle.
$\triangle$,	triangle.
▱,	parallelogram.
▭,	rectangle.
$\odot$,	circle.
$\therefore$,	therefore.
$\equiv$,	is identically equal to.
$\rightarrow$,	approaches as limit.
i. m. b.,	is measured by.

Any symbol representing a noun is converted into the plural by affixing the letter *s*; thus $\angle s$ means angles.

Abbreviations

Adj.,	adjacent.
Alt.,	alternate.
Ax.,	axiom.
Comp.,	complementary.
Con.,	conclusion.
Cong.,	congruent.
Const.,	construction.
Cor.,	corollary.
Corres.,	corresponding.
Def.,	definition.
Ex.,	exercise.
Ext.,	exterior.
Hyp.,	hypothesis.
Int.,	interior.
Rect.,	rectangle.
Rt.,	right.
St.,	straight.
Supp.,	supplementary.

SOLID GEOMETRY

BOOK VI

LINES AND PLANES — POLYEDRAL ANGLES

406. The difference between plane and solid geometry. In plane geometry, the figures consisted of points and lines which lie in only one plane (§ 2), — like the plane of the blackboard. Recall that no definition of plane was given (§ 1) or of point or straight line. (See Note 1, p. 24.)

In solid geometry, the figures do not all lie in one plane. For this reason, some of the theorems of plane geometry are not true in solid geometry.

We shall study not only figures consisting of points and lines, but also new figures called surfaces.

407. Surfaces. No satisfactory elementary definition of a surface, in general, can be given. The surface of a physical object is that part of the object which can, in general, be touched; it separates the object from surrounding space.

The surface of a pond on a calm day is a *plane surface.*

The surface of a croquet ball is a *spherical surface.*

The surface of a "round" column is a *cylindrical surface.*

Ex. 1. If two points on the surface of a ball were joined by a straight line, would the line lie on the surface?

Ex. 2. Are there any two points on the surface of a ball such that the straight line through them lies upon the surface of the ball?

Ex. 3. Are there two points on the surface of a cylindrical column such that the straight line joining them lies on the surface of the column?

Ex. 4. Does the straight line joining every pair of points on the surface of a cylindrical column lie upon the surface of the column?

Ex. 5. Give other examples of plane and curved surfaces.

408. Three postulates (§ 50) **of the plane** are suggested by Exercises 1–4, and others like them.

Postulate 1. *If two points of a straight line lie in a plane, every point of the line lies in the plane,* — or the line lies wholly in the plane.

Postulate 2. *Three non-collinear* points lie in one and only one plane;* or three non-collinear points *determine* a plane.

Postulate 3. *If two planes have one common point, then they have at least a second common point.*

409. Observe the second expression of Postulate 2. When we say that *a figure is determined by* some points, or lines, etc., we mean that it is the one and only figure of its kind which does contain the given points, or lines, etc.

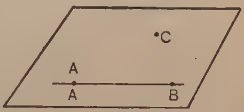
Thus, *two points determine a straight line,* means that one and only one straight line contains the two given points.

410. Immediate consequences of the postulates of the plane.

Cor. 1. *A plane extends infinitely far.*

For, the straight line determined by two points of a plane lies wholly in the plane, by Postulate 1. Since this line extends infinitely far in two directions, then the plane must also.

Cor. 2. *A plane is determined by a straight line and a point not on the line.*



For any two points, *A* and *B*, of the line and the given point *C*, not on the line, determine a plane by Postulate 2.

Since *A* and *B* are in this plane, then *AB* must also be in the plane, by Postulate 1.

Ex. 6. Do two straight lines drawn at random necessarily lie in a plane? Illustrate by holding two pencils.

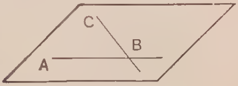
Ex. 7. Do four points usually lie in a plane? Select four in the schoolroom that do and four that do not.

Ex. 8. Are two straight lines in space parallel if they do not meet no matter how far they are extended? (See § 88.)

* Points are non-collinear if they do not lie in a straight line.

Cor. 3. *A plane is determined by two intersecting straight lines.*

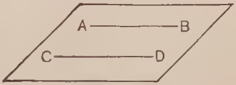
For a plane is determined by AB, and any point C of BC, excepting B, the point of intersection of AB and BC, by Cor. 2, § 410.



AB is in this plane. Since B and C are in the plane, then, by Postulate 1, BC also is in the plane.

Cor. 4. *A plane is determined by two parallel lines.*

Parallels AB and CD already lie in one plane, by definition. (§ 88.)



They cannot lie in more than one plane, for if they did, points A, B, and C would lie in more than one plane, which is impossible by Postulate 2.

411. Points and lines lying in the same plane are said to be **co-planar.**

Points lying on the same straight line are said to be **collinear.**

Ex. 9. Why is a tripod used as mounting for a camera or a surveyor's instrument?

Ex. 10. Why does a stool with three legs stand firmly whereas one with four legs cannot always be made to stand firmly?

Ex. 11. Prove that a plane and a straight line not lying in the plane can have only one common point.

Ex. 12. In how many different planes can one straight line lie?

Ex. 13. How many different planes are determined:

(*a*) By three concurrent (§ 162) lines which do not all lie in one plane?

(*b*) By three parallel lines which do not all lie in one plane?

(*c*) By two intersecting lines and a point which does not lie in their plane?

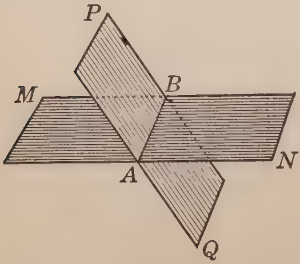
(*d*) By four points, no three of which are collinear and which do not all lie in one plane?

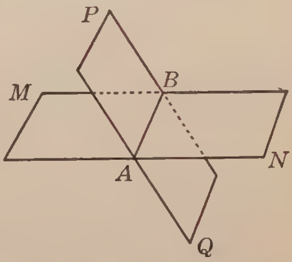
Ex. 14. Prove that two parallels and any transversal of them are co-planar.

412. The intersection of two surfaces or of a surface and a line consists of all points common to the surfaces, or to the surface and the line.

PROPOSITION I. THEOREM

413. *If two planes intersect, the intersection is a straight line.*





Hypothesis. A and B are two points common to the two planes MN and PQ.

Conclusion. The intersection of MN and PQ is a straight line.

Proof:

	Statements	Reasons
1.	A and B are in plane MN.	1. By the hyp.
2.	$\therefore AB$ is in MN.	2. § 408; Post. 1.
3.	Similarly AB is in PQ.	3. Give proof.
4.	If a point outside AB were in MN and PQ, PQ and MN would coincide.	4. § 410; Cor. 2.
5.	But MN and PQ are *two* planes.	5. By the hyp.
6.	$\therefore$ the intersection of MN and PQ is straight line AB.	6. § 412.

Note. *Dotted lines* will be used to indicate lines which are hidden by planes. Lines made by *short dashes* will be used to indicate construction lines.

Pupils should draw unshaded figures like the right hand one above.

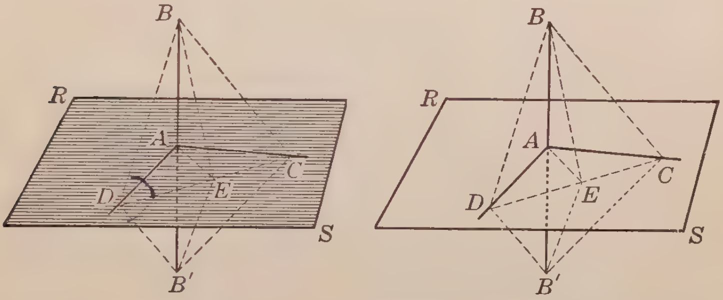
414. If a straight line intersects a plane, the point of intersection of the line and the plane is the **foot of the line.**

A straight line can intersect a plane in only one point.

415. A line is perpendicular to a plane if it is perpendicular to every line in the plane passing through its foot. The plane is also perpendicular to the line.

* PROPOSITION II. THEOREM

416. *If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to their plane.*



Hypothesis. AD and AC, intersecting, determine plane RS.
$BA \perp AD$; $BA \perp AC$.

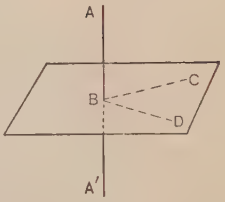
Conclusion. $BA \perp$ plane RS.

Proof:

	STATEMENTS	REASONS
1.	In RS, draw AE, any other line through A. Draw DEC, intersecting AD, AE, and AC at D, E, and C, respectively.	1. These are all possible constructions.
2.	Extend BA to B', making $AB' = BA$. Draw BD, BE, BC, $B'D$, $B'E$, and $B'C$.	2. All possible constructions.
3.	$\therefore BD = B'D$, and $BC = B'C$.	3. Prove it. [illegible]
4.	$\therefore \triangle BDC \cong \triangle B'DC$.	4. Why? S.S.S.
5.	$\therefore \angle BDE = \angle B'DE$.	5. Why? ≅ △
6.	$\therefore \triangle BDE \cong \triangle B'DE$.	6. Prove it. [illegible]
7.	$BE = B'E$.	7. Why? ≅ △
8.	But $BA = B'A$.	8. Step 2.
9.	$\therefore AE$ is a $\perp$-bis. of BB'.	9. § 77.
10.	$\therefore BA \perp AE$.	10. Why? ⑨
11.	$\therefore BA \perp$ plane RS.	11. § 415.

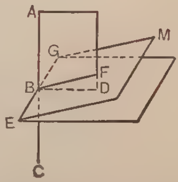
417. Cor. 1. *Through a point on a given line, a plane can be passed perpendicular to the line.*

Plan. Draw BC and BD, any two different perpendiculars to AA' at B. Prove plane BCD $\perp$ to AA'.



Note. *Through a point in a given line, only one plane can be passed perpendicular to the line.*

If planes ME and RE were both $\perp$ to AC at B, a plane AD through AC would intersect ME and RE in two lines, BF and BD, each $\perp$ to AC at B. But in a plane (AD) only one line can be drawn $\perp$ to a given line at a point in the line.

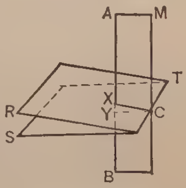


418. Cor. 2. *Through a point not in a given line, a plane can be passed perpendicular to the line.* (Figure of § 417.)

Plan. Draw $CB \perp$ to AA' from C; draw $BD \perp AA'$ at B, in some plane other than plane $AA'C$. Prove plane $CBD \perp$ to AA'.

Note. *Through a point not in a given line, only one plane can be passed perpendicular to the line.*

If planes RT and ST, through C, were both $\perp$ to AB, the plane ABC would intersect RT and ST in lines XC and YC, each $\perp$ to AB. But in a plane (ABC) only one $\perp$ can be drawn to a given line (AB) from a point (C) not in the line.



419. *If two oblique segments, drawn to a plane from a point in a perpendicular to the plane, cut off equal segments from the foot of the perpendicular, they are equal.* (Prove it.)

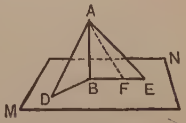
420. *If two oblique segments, drawn to a plane from a point in a perpendicular to the plane, cut off unequal segments from the foot of the perpendicular, the one cutting off the greater segment is the greater.* (Prove it.)

Hyp. $AB \perp$ plane MN; BE and BD are in MN.

$BE > BD$.

Con. $AE > AD$.

Plan. Take $BF = BD$, and draw AF.



Ex. 15. If a line is perpendicular to a line of a plane, is it perpendicular to the plane?

Ex. 16. State and prove the converse of § 419.

* PROPOSITION III. THEOREM

421. *Through a point in a plane, one and only one straight line can be drawn perpendicular to the plane.*

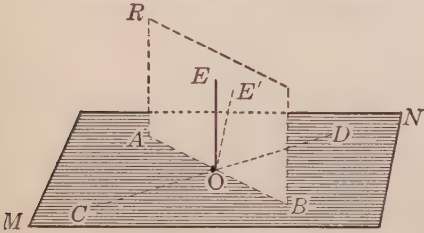


FIG. 1

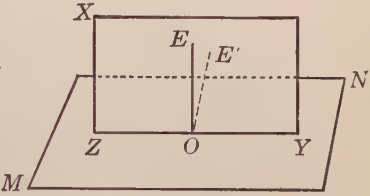


FIG. 2

Hypothesis. Point O lies in plane MN. (Fig. 1.)

Conclusion. (I) One perpendicular, and (II) only one perpendicular can be drawn to MN through O.

I. Construction:

		REASONS
1.	In plane MN, draw any line CD through O.	1. Possible.
2.	Through O, pass plane $RB \perp CD$.	2. § 417.
3.	Let RB intersect MN in line AB.	3. § 413.
4.	In RB, through O, draw $EO \perp AB$.	4. § 80.

Statement: $EO \perp$ plane MN.

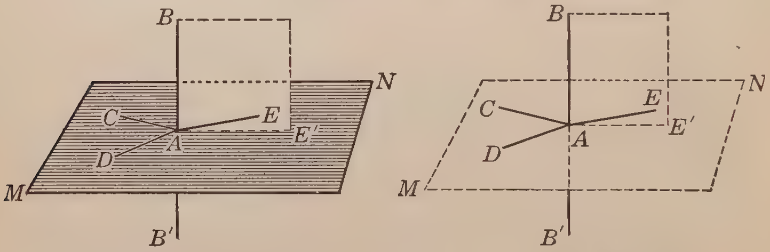
Proof:	STATEMENTS	REASONS
1.	Since $CD \perp RB$, $EO \perp CD$.	1. § 415.
2.	But $EO \perp AB$.	2. Const.
3.	$\therefore EO \perp$ plane MN.	

Proof of II:	(Fig. 2.) STATEMENTS	REASONS
1.	Suppose EO and $E'O$ were both $\perp$ to MN.	1. See § 93.
2.	EO and $E'O$ determine a plane XY.	2. Why?
3.	Plane XY intersects plane MN in line ZY.	3. Why?
4.	$\therefore EO \perp ZY$ and $E'O \perp ZY$.	4. § 415.
5.	But this is impossible.	5. § 81.
6.	$\therefore EO$ is the only $\perp$ to MN through O.	6. See § 93.

Ex. 17. Illustrate the construction described in this proposition, using for O a point on your desk, and for RB a sheet of cardboard.

PROPOSITION IV. THEOREM

422. *All the perpendiculars to a straight line through a point in the line lie in the plane which is perpendicular to the line through the given point.*



Hypothesis. Plane $MN \perp AB$, through point A.
AE is any line $\perp AB$ through A.

Conclusion. AE lies in plane MN.

Proof:

	STATEMENTS	REASONS
1.	AB and AE determine a plane ABE.	1. Why?
2.	Plane ABE intersects MN in a line AE'.	2. Why?
3.	$\therefore AB \perp AE'$, or $AE' \perp AB$.	3. § 415.
4.	But $AE \perp AB$.	4. Why?
5.	$\therefore$ in plane $ABEE'$, AE coincides with AE'.	5. § 81.
6.	$\therefore AE$ lies in plane MN.	6. Why?

Ex. 18. Each of three concurrent straight lines is perpendicular to the other two. Prove that each is perpendicular to the plane of the other two.

Ex. 19. If a circle be drawn in a plane and a perpendicular be erected at its center, any point in this perpendicular is equidistant from all points of the circle.

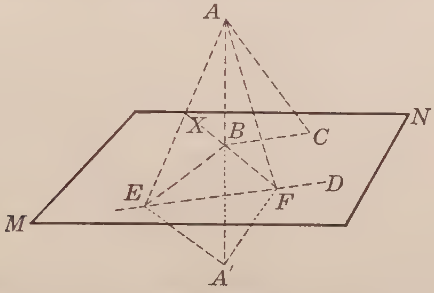
Ex. 20. A line segment of fixed length, having one extremity at a fixed point not on a given plane, has the other extremity on the plane. What is the locus of the extremity which lies on the plane?

Ex. 21. If the spokes of a wheel are actually perpendicular to the axle running through the hub, where does the rim of the wheel lie?

Ex. 22. If the hands of a clock are perpendicular to the shaft on which they are placed, do they revolve in planes?

* PROPOSITION V. THEOREM

423. *One and only one straight line can be drawn perpendicular to a plane, from a point not in the plane.*



Hypothesis. Point A lies outside plane MN.

Conclusion. (I) One and (II) only one straight line can be drawn $\perp$ to MN through A.

I. Construction:	REASONS
1. Draw DE, any straight line in plane MN.	1. Possible.
2. From A, draw $AF \perp DE$, meeting it at F.	2. § 82.
3. In MN, draw $XF \perp DE$, at F.	3. § 80.
4. In plane AFX, draw $AB \perp XF$.	4. § 82.

Statement : $AB \perp$ to plane MN.

Proof: STATEMENTS	REASONS
1. Extend AB to A' so that $BA' = AB$. Draw EA, EB, and EA' ; also FA'.	1. Possible construction.
2. $DE \perp$ plane $AA'F$ and $DE \perp A'F$.	2. § 416, § 415.
3. $\triangle AFE \cong \triangle A'FE$, and $AE = A'E$.	3. Prove it.
4. $\therefore EB \perp AA'$, and $AA' \perp BE$.	4. Prove it.
5. $\therefore ABA' \perp$ plane MN.	5. § 416.

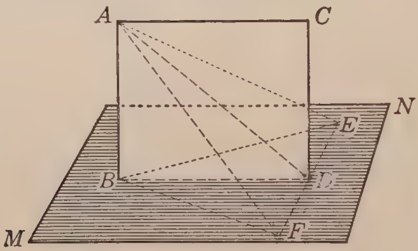
Proof of II : If AC *also* were $\perp$ to MN, then AC and AB would both be $\perp$ to BC, and this is impossible.

424. Cor. *The perpendicular to a plane from a point not in it is the shortest segment from the point to the plane.*

425. The **distance from a point to a plane** is the length of the perpendicular segment from the point to the plane.

* PROPOSITION VI. THEOREM

426. *If two lines are perpendicular to the same plane, they are parallel.*



Hypothesis. $AB \perp MN$ at B; $CD \perp MN$ at D.

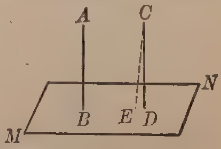
Conclusion. $AB \parallel CD$.

Plan. Prove AB and CD co-planar and $\perp$ to the same line.

Proof:

	STATEMENTS	REASONS
1.	Draw BD and AD. In MN, draw $FE \perp BD$. Take $DF = DE$. Draw BF, BE, AF, and AE.	1. Possible.
2.	$BF = BE$; and $\therefore AF = AE$.	2. Prove it.
3.	$\therefore AD \perp FE$.	3. Prove it.
4.	But $CD \perp FE$; and $BD \perp FE$.	4. Why?
5.	$\therefore CD$, AD, and BD lie in a plane $ABDC$.	5. § 422.
6.	$\therefore AB$ and CD are co-planar.	6. § 408, I.
7.	$\therefore AB \parallel CD$.	7. Prove it.

427. Cor. 1. *If one of two parallel lines is perpendicular to a plane, the other is also.*

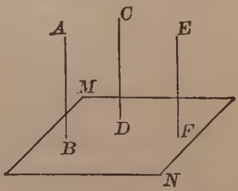


Hyp. $AB \parallel CD$. $AB \perp$ plane MN.

Con. $CD \perp$ plane MN.

Plan. Assume $CE \perp MN$.

428. Cor. 2. *If each of two straight lines is parallel to a third line, they are parallel to each other.*



Hyp. $AB \parallel CD$. $EF \parallel CD$.

Con. $AB \parallel EF$.

Plan. Assume $MN \perp CD$.

REVIEW EXERCISES

Ex. 23. State three theorems by which to determine a plane.

Ex. 24. State the three postulates of the plane.

Ex. 25. What is meant by the intersection of two surfaces?

Ex. 26. How many lines of intersection are determined by four planes which are so located that no three of them pass through the same straight line, if each intersects each of the others?

Ex. 27. (*a*) Prove that a triangle lies in a plane.

(*b*) Does a four-sided figure necessarily lie in a plane?

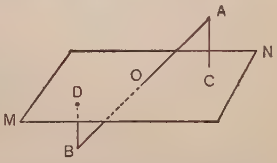
Ex. 28. When is a line perpendicular to a plane?

Ex. 29. Where do all straight lines lie which are perpendicular to the same straight line at the same point H?

Ex. 30. What is true about two straight lines in the same plane which are both perpendicular to the same straight line?

Ex. 31. Is the same conclusion (as in Ex. 30) true if the lines do not lie in the same plane? (Illustrate with pencils.)

Ex. 32. If two points lie on opposite sides of a plane and are equidistant from it, the segment joining them is bisected by the plane.



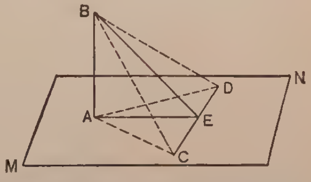
Ex. 33. If two oblique lines, drawn to a plane from a point in a perpendicular to the plane, cut off equal segments from the foot of the perpendicular, they make equal angles with the perpendicular.

Ex. 34. If through the foot of a perpendicular to a plane a line be drawn perpendicular to any line in the plane, then the line drawn from its intersection with the line in the plane to any point on the given perpendicular is also perpendicular to the given line in the plane.

Hyp. $AB \perp$ plane MN;
$AE \perp CD$ at E.

Con. $BE \perp CD$.

Plan. Take $ED = EC$. Prove $BD = BC$ by first proving $AD = AC$.



Ex. 35. Recall the definition of converse given in § 104. Make a converse of Ex. 34 by interchanging $AE \perp CD$ and $BE \perp CD$. Try to prove the resulting theorem.

Ex. 36. How does a carpenter use the principle of § 416 when placing a corner post so that it will be "plumb," or perpendicular to a level plane?

Note. Additional Exercises 1 to 5, p. 357, can be done now.

PARALLEL LINES AND PLANES

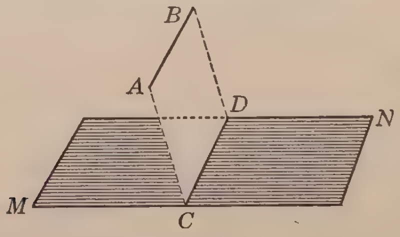
429. **A straight line and a plane are parallel** if they do not meet however far they may be extended.

Two planes are parallel if they do not meet however far they may be extended.

Two straight lines are skew if they do not lie in the same plane and do not meet however far they may be extended.

PROPOSITION VII. THEOREM

430. *If a line outside a plane is parallel to a line of the plane, it is parallel to the plane.*



Hypothesis. $AB \parallel CD$.

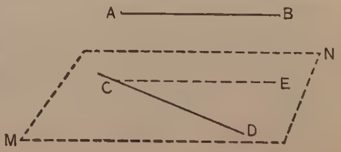
Plane MN contains CD but not AB.

Conclusion. $AB \parallel$ plane MN.

Proof:

	STATEMENTS	REASONS
1.	AB and CD lie in a plane AD.	1. § 88.
2.	CD is the intersection of AD and MN.	2. Why?
3.	If AB were to intersect MN, the intersection point would be in AD and in MN.	3. Why?
4.	$\therefore$ AB would intersect CD.	4. § 412.
5.	But $AB \parallel CD$.	5. Why?
6.	$\therefore$ AB cannot meet MN, or $AB \parallel MN$.	6. See § 93.

431. **Cor. 1.** *Through either of two skew straight lines, one and only one plane can be passed parallel to the other line.*

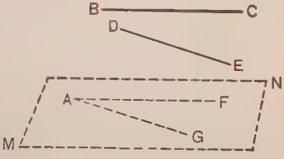


Prove a plane can be passed through CD, parallel to AB.

Plan. Draw $CE \parallel AB$.

432. Cor. 2. *Through a given point not in either line, one and only one plane can be passed parallel to each of two given skew straight lines.*

Prove a plane can be passed through *A* parallel to *BC* and *DE*.



Plan. Draw $AG \parallel DE$, and $AF \parallel BC$. Prove plane *AGF* is $\parallel$ to *BC* and *DE*.

Ex. 37. Using three pencils to represent the lines *AB*, *CD*, and *CE*, illustrate the construction suggested in § 431.

Ex. 38. Let two pupils, each using two pencils, represent the lines *BC*, *DE*, *AG*, and *AF*, and illustrate the construction of § 432.

Ex. 39. Through a given point not on a given line, an infinity of planes can be passed parallel to the given line.

Ex. 40. Through a given point not on a plane an infinity of lines can be drawn parallel to the plane.

Ex. 41. Through a given point, a line can be drawn parallel to each of two intersecting planes.

Ex. 42. Does the upper moulding of your front blackboard appear to be parallel to the intersection of the front wall and ceiling? If it is, what is true about the moulding and the ceiling?

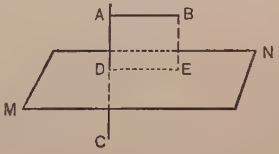
Ex. 43. Prove that any edge of a crayon box is parallel to each of two faces of the box.

Ex. 44. If two planes are parallel, every line of one of them is parallel to the other.

Suggestion. Use the indirect method of proof.

Ex. 45. You know that two lines are parallel if they are parallel to the same straight line. Are two straight lines necessarily parallel if they are both parallel to the same plane?

Ex. 46. Prove that a straight line and a plane, both perpendicular to the same straight line, are parallel.



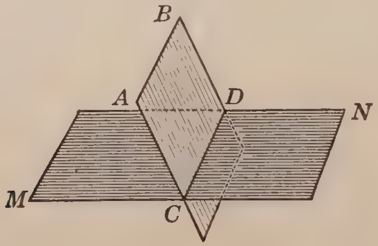
Hyp. $AB \perp AC$; plane $MN \perp AC$.

Con. $AB \parallel$ plane *MN*.

Plan. Let the plane determined by *AB* and *AC* intersect *MN* in line *DE*.

PROPOSITION VIII. THEOREM

433. *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*



Hypothesis. $AB \parallel$ plane MN.

Plane BC, through AB, intersects MN in CD.

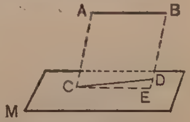
Conclusion. $AB \parallel CD$.

Plan. Prove AB cannot intersect CD.

Proof:

	Statements	Reasons
1.	AB and CD lie in plane $ABDC$.	1. Hypothesis.
2.	If AB were to intersect CD, it would necessarily intersect MN.	2. Why?
3.	But $AB \parallel MN$.	3. Why?
4.	$\therefore AB$ cannot intersect CD.	4. See § 93.
5.	$\therefore AB \parallel CD$.	5. Why?

434. Cor. *If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*



Hyp. $AB \parallel$ plane MN.
C is any pt. in MN.
$CD \parallel AB$.

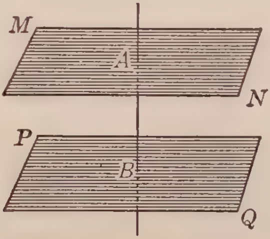
Con. CD lies in MN.

Proof. 1. The plane determined by AB and C intersects MN in a line CE, through C, parallel to AB.

2. But CD, through C, $\parallel AB$. Why?
3. CD and CE coincide. Why?
4. CD lies in plane MN. Why?

PROPOSITION IX. THEOREM

435. *Two planes perpendicular to the same straight line are parallel.*



Hypothesis. Planes MN and PQ are $\perp$ to AB.

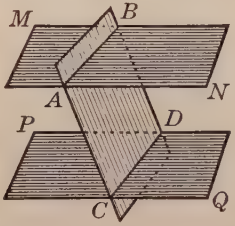
Conclusion. $MN \parallel PQ$.

Plan. Prove it by the indirect method, using § 418.

PROPOSITION X. THEOREM

[illegible]

436. *If two parallel planes are cut by a third plane, the intersections are parallel.*



Hypothesis. Plane $MN \parallel$ plane PQ.

Plane AD intersects MN in AB and PQ in CD.

Conclusion. $AB \parallel CD$.

Plan. Use § 88. Use the indirect proof, § 93.

Ex. 47. Prove parallel segments between parallel planes are equal.

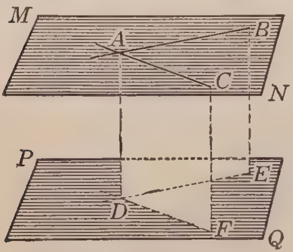
Ex. 48. If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

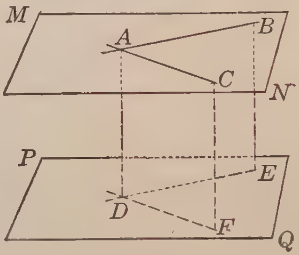
Ex. 49. Prove that a line parallel to a plane is everywhere equidistant from it.

Ex. 50. If two points are equidistant from a plane, and on the same side of it, they determine a line parallel to the plane.

PROPOSITION XI. THEOREM

437. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*





Hypothesis. Plane $MN \parallel$ plane PQ. $AD \perp PQ$.

Conclusion. $AD \perp MN$.

Plan. Prove $AD \perp$ two intersecting lines in MN.

Proof:

STATEMENTS	REASONS
1. Through AD, pass two planes intersecting MN in AB and AC, and PQ in DE and DF, respectively.	1. Possible construction.
2. $AB \parallel DE$; and $AC \parallel DF$.	2. § 436.
3. $AD \perp DE$; and $AD \perp DF$.	3. Why?

(Complete the proof, using § 101 and § 435.)

PROPOSITION XII. THEOREM

438. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*

Hypothesis. $AB \parallel$ plane PQ; $AC \parallel$ plane PQ.
AB and AC determine plane MN.

Conclusion. $MN \parallel PQ$.

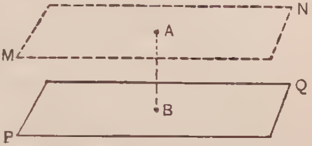
Plan. Prove the two planes $\perp$ to the same st. line.

Proof:

STATEMENTS	REASONS
1. From A, draw $AD \perp PQ$.	1. § 423.
2. The plane of AC and AD intersects PQ in DF, $\parallel$ to AC. Similarly $DE \parallel AB$.	2. § 433.

(Now prove: (*a*) $AD \perp DF$ and $AD \perp DE$; (*b*) $AD \perp MN$; (*c*) $MN \parallel PQ$.)

439. **The distance between two parallel planes** is the length of the segment perpendicular to and between them.



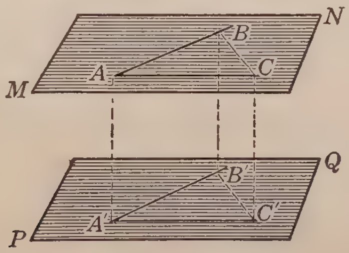
440. *Two parallel planes are everywhere equidistant.*

441. *Through a given point not in a given plane, one and only one plane can be passed parallel to the given plane.*

Plan. Draw $AB \perp PQ$ and $MN \perp AB$, through A. To prove the second part, use § 93, § 437, and Note, § 417.

* PROPOSITION XIII. THEOREM

442. *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*



Hypothesis. $\angle BAC$ is in plane MN; $\angle B'A'C'$ is in plane PQ. $AB \parallel A'B'$. $AC \parallel A'C'$.

Conclusion. $\angle BAC = \angle B'A'C'$; $MN \parallel PQ$.

Plan. Prove $\angle A$ and $\angle A'$ corres. parts of cong. $\triangle$s.

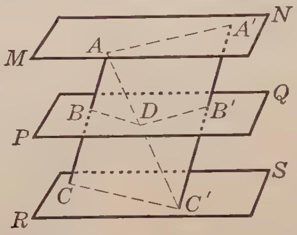
Proof:

STATEMENTS	REASONS
1. $AB \parallel$ plane PQ; also $AC \parallel$ plane PQ.	1. § 430.
2. $\therefore MN \parallel PQ$.	2. § 438.
3. Make $A'B' = AB$ and $A'C' = AC$. Draw AA', BB', CC', BC, and $B'C'$.	3. All possible constructions.
4. $BB' = AA'$; and $BB' \parallel AA'$. Similarly $CC' = AA'$; and $CC' \parallel AA'$.	4. Prove it.

(Now prove: (*a*) $BB'C'C$ is a ▱; (*b*) $\triangle ABC \cong \triangle A'B'C'$.)

PROPOSITION XIV. THEOREM

443. *If two straight lines are cut by three or more parallel planes, the corresponding segments are proportional.*



Hypothesis. Planes MN, PQ, and RS are parallel.
AC intersects the planes at A, B, and C, respectively.
$A'C'$ intersects the planes at A', B', and C', respectively.

Conclusion. $$\frac{AB}{BC}=\frac{A'B'}{B'C'}$$

Plan. Prove $\frac{AB}{BC}$ and $\frac{A'B'}{B'C'}$ = to the same ratio.

Proof:

STATEMENTS	REASONS
1. Let AC' cut PQ at D. Then plane ACC' intersects RS in CC', and PQ in BD; similarly plane $AC'A'$ intersects PQ in DB', and MN in AA'.	1. Cor. 3, § 410. § 413.
2. $BD \parallel CC'$; and $DB' \parallel AA'$.	2. Why?

(Complete the proof, using § 262, and Ax. 1.)

Ex. 51. If two planes are parallel to a third plane, they are parallel to each other.

Ex. 52. From a parallel to a plane, two parallel segments are drawn to and terminated by the plane. Prove the segments equal.

Ex. 53. Through a given line which is parallel to a given plane, a number of planes are passed, intersecting the given plane. Prove that all the lines of intersection are parallel.

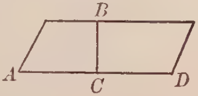
Ex. 54. If one of two parallel lines is parallel to a plane, the other is also, unless it lies in the plane.

Ex. 55. Prove that the segments joining in order the midpoints of the sides of a skew quadrilateral form a parallelogram.

Note. Additional exercises 6 to 11, p. 457, can be done now.

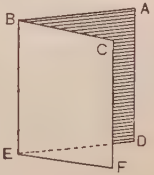
DIEDRAL ANGLES

444. It is evident that a straight line divides a plane into two parts. A part of the plane, like BCD, is called a **half-plane.** BC is called the **edge** of the half-plane.



445. A **diedral angle** is the figure formed by two half-planes which have a common edge.

The common edge is the **edge** and the two half-planes are the **faces** of the diedral angle.



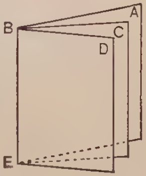
Thus, BF and BD form the diedral angle whose edge is BE, and whose faces are FBE and DBE.

The diedral angle may be read:

diedral $\angle BE$, or diedral $\angle ABEC$.

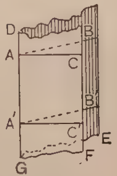
446. Two diedral angles are **adjacent** when they have the same edge and a common face between them : as $\angle ABEC$ and $\angle CBED$.

Two diedral angles are **vertical** when the faces of one are the extensions through the edge of the faces of the other.



447. A plane angle of a diedral angle is formed by two rays, one in each face, drawn perpendicular to the edge at the same point.

Thus, if AB and AC be drawn in faces DE and DF respectively, perpendicular to DG at A, $\angle BAC$ is a plane angle of the diedral angle DG.



448. *All plane angles of a given diedral angle are equal.*

Prove $\angle BAC = \angle B'A'C'$.

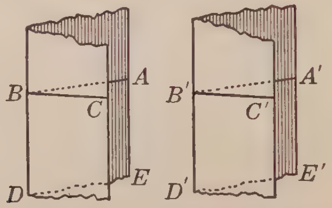
449. *A plane perpendicular to the edge of a diedral angle intersects the faces of the angle in lines which form a plane angle of the diedral angle.*

If a plane intersects $\angle DG$ in AB and AC, so that it is perpendicular to edge DG, prove $AB \perp DG$ and $AC \perp DG$.

450. Two diedral angles are equal if they can be made to coincide.

PROPOSITION XV. THEOREM

451. *Two diedral angles are equal if their plane angles are equal.*



Hypothesis. $\angle CBA$ and $\angle C'B'A'$ are the plane $\measuredangle$ of diedral $\measuredangle$ BD and $B'D'$ respectively; $\angle CBA = \angle C'B'A'$.

Conclusion. $\angle BD = \angle B'D'$.

Plan. Prove that the $\measuredangle$ can be made to coincide.

Proof:

STATEMENTS	REASONS
1. Place $\angle BD$ on $\angle B'D'$ so that $\angle CBA$ coincides with its equal, $\angle C'B'A'$.	1. § 60.
2. $BD \perp$ plane CAB; $BD \perp$ plane $C'A'B'$.	2. Why?
3. $\therefore BD$ coincides with $B'D'$.	3. § 421.
4. $\therefore A'D'$ coincides with AD, and $C'D'$ with CD.	4. Why?
5. $\therefore \angle B'D' = \angle BD$.	5. Why?

452. Cor. 1. *If two diedral angles are equal, their plane angles are equal.*

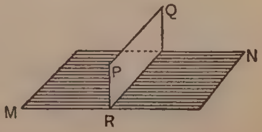
Plan. Make the $\measuredangle$ coincide; then form their plane $\angle$.

453. Cor. 2. *If two planes intersect, vertical diedral angles are equal.*

Plan. Compare their plane angles.

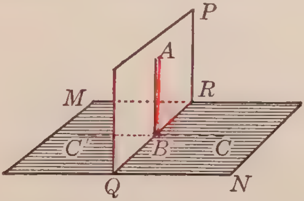
454. A diedral angle is right, acute, or obtuse according as its plane angle is right, acute, or obtuse. Two **diedral angles are supplementary or complementary** according as their plane angles are supplementary or complementary.

455. Two planes are perpendicular if the diedral angles formed are right diedral angles.



* PROPOSITION XVI. THEOREM

456. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Hypothesis. Plane $PQ \perp$ plane MN. PQ intersects MN in QR. Line AB in $PQ \perp QR$.

Conclusion. $AB \perp MN$.

Proof:

	Statements	Reasons
1.	Draw $C'BC$ in MN, $\perp$ to QR at B.	1. Possible.
2.	But $AB \perp QR$ at B.	2. Why?
3.	$\therefore \angle ABC$ is the plane $\angle$ of $\angle PQRN$.	3. Why?
4.	But $\angle PQRN$ is a rt. diedral $\angle$.	4. Why?

(Complete the proof, using § 454 and § 416.)

457. Cor. 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection lies in the other.*

Hyp. Plane $PQ \perp$ plane MN, intersecting it in QR.
AB, drawn from any pt. B of QR, $\perp MN$.

Con. AB lies in PQ.

Plan. Prove that a line in PQ, $\perp$ to QR at B, coincides with AB.

458. Cor. 2. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other lies in the other.*

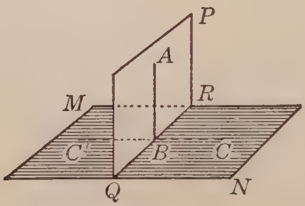
Hyp. Plane $PQ \perp$ plane MN, intersecting it in QR;
AB, drawn from any pt. A of PQ, $\perp MN$.

Con. AB must lie in PQ.

Plan. Prove that a $\perp$ from A to QR must coincide with AB.

* PROPOSITION XVII. THEOREM

459. *If a straight line is perpendicular to a plane, every plane which contains the line is perpendicular to the plane.*



Hypothesis. $AB \perp$ plane MN, at B.

PQ, any plane which contains AB, intersects MN in QBR.

Conclusion. $PQ \perp MN$.

Plan. Prove $\angle PQRN$ is a rt. diedral $\angle$.

Proof:

	STATEMENTS	REASONS
1.	In MN, draw $C'BC$ through B, $\perp QR$.	1. § 80.
2.	But $AB \perp QR$, through B.	2. Why?
3.	$\therefore \angle ABC$ is the plane $\angle$ of $\angle PQRN$.	3. Why?

(Complete the proof, using § 415, § 454, and § 455.)

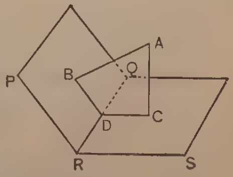
Ex. 56. (*a*) Prove that a plane can be passed through a point perpendicular to a given plane. (*b*) How many such planes can be drawn?

Ex. 57. Prove that a plane perpendicular to the edge of a diedral angle is perpendicular to the faces of the angle.

Ex. 58. Prove that a plane can be drawn bisecting a diedral angle.

Ex. 59. If one plane meets another plane, the adjacent diedral angles formed are supplementary.

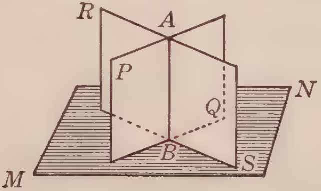
Ex. 60. If perpendiculars are drawn to the faces of a diedral angle from any point within the angle, they lie in a plane perpendicular to the edge of the diedral angle and form an angle which is the supplement of the plane angle of the diedral angle.



Plan. Use the sum of the angles of a plane quadrilateral.

* PROPOSITION XVIII. THEOREM

460. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*



Hypothesis. Plane $MN \perp$ planes RS and PQ.
RS intersects PQ in line AB.

Conclusion. $MN \perp AB$.

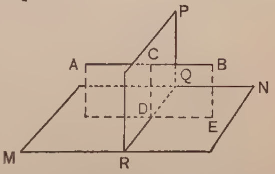
Plan. Prove a $\perp$ to MN from A coincides with AB. (§ 458.)

Ex. 61. Are two planes which are perpendicular to the same plane necessarily parallel?

Ex. 62. If a plane is perpendicular to one of two parallel planes, it is perpendicular to the other also.

Ex. 63. If a plane is perpendicular to a line of a plane, it is perpendicular to the plane.

Ex. 64. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

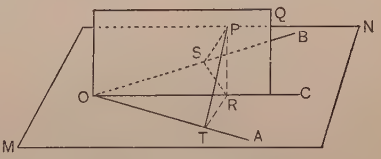


Hyp. $AB \parallel$ plane MN.
Plane $PR \perp AB$ at C, intersecting MN in RQ.

Con. $PR \perp MN$.

Plan. 1. Draw line CD in $PR \perp QR$.
2. Let the plane determined by CB and CD intersect MN in DE.
3. Prove $CD \perp MN$.

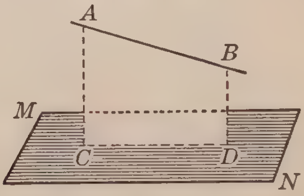
Ex. 65. Prove that any point in the plane through the bisector of an angle and perpendicular to the plane of the angle is equidistant from the sides of the angle.



Plan. 1. Draw $PR \perp OC$, $RT \perp OA$, and $RS \perp OB$. Draw PT and PS.
2. Prove $PR \perp MN$ (§ 456), $PT \perp OA$ (Ex. 34), $PS \perp OB$.

PROPOSITION XIX. THEOREM

461. *Through a given straight line not perpendicular to a given plane, one and only one plane can be passed perpendicular to the given plane.*



Hypothesis. AB is not $\perp$ to plane MN.

Conclusion. One and only one plane can be passed through AB $\perp$ to MN.

Proof: Statements	Reasons
1. Draw $AC \perp$ to MN from A.	1. § 423.
2. AC and AB determine a plane $\perp$ to MN.	2. Prove it.
3. If two planes through AB were $\perp$ to MN, then AB, their intersection, would also be $\perp$ to MN.	3. Why.
4. But AB is not $\perp$ to MN.	4. Why?
5. $\therefore$ only one plane can be passed through $AB \perp$ to MN.	5. See § 93.

Note 1. If AB lies in MN, the theorem is still true. Why?

Note 2. If $AB \perp MN$, an infinity of planes can be passed through $AB \perp$ to MN. Why?

Ex. 66. Given a line and a point not on that line, — both lying in a given plane. Prove that a plane can be passed through the given point, perpendicular to the given plane, and parallel to the given line.

Ex. 67. Given two parallel lines whose plane is not perpendicular to a second given plane. Prove that planes passed through these lines, perpendicular to the second given plane, are parallel.

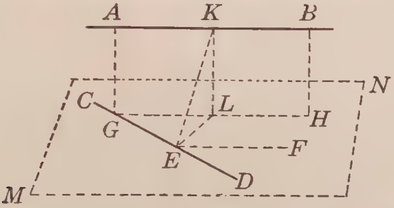
Suggestion. Use § 461 and § 442.

Ex. 68. If a plane and a line not in it are both perpendicular to a second plane, then the first plane and the line are parallel.

Suggestion. In the first plane, draw a line perpendicular to the intersection of the two planes.

PROPOSITION XX. THEOREM

462. *Between two skew straight lines, one and only one common perpendicular can be drawn.*



Hypothesis. AB and CD are two skew straight lines.

Conclusion. (I) One and (II) only one common $\perp$ to AB and CD can be drawn.

Proof of I:

	STATEMENTS	REASONS
1.	Through CD, pass plane $MN \parallel$ to AB.	1. § 431.
2.	Through AB, pass plane $AH \perp$ to MN, intersecting MN in line GH.	2. § 461.
3.	$\therefore GH \parallel AB$.	3. § 433.
4.	If $GH \parallel CD$, then $AB \parallel CD$.	4. § 428.
5.	But AB is not $\parallel$ to CD.	5. Hyp.
6.	$\therefore GH$ intersects CD at a point G.	6. § 93.
7.	In plane AH, draw $AG \perp$ to GH, at G.	7. § 80.
8.	$\therefore AG \perp MN$.	8. § 456.
9.	$\therefore AG \perp CD$; also $AG \perp AB$.	9. Prove it.

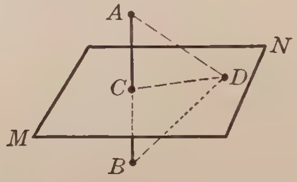
Proof of II:

	STATEMENTS	REASONS
1.	Assume $KE \perp CD$ and also $KE \perp AB$.	1. See § 93.
2.	Draw $EF \parallel$ to AB. Then EF lies in MN.	2. § 434.
3.	$\therefore KE \perp EF$, and $KE \perp MN$.	3. Prove it.
4.	Draw KL in plane $AH \perp$ to GH.	4. § 82.
5.	$\therefore KL \perp MN$.	5. § 456.
6.	But KE and KL cannot both be $\perp$ to MN.	6. Why?
7.	$\therefore KE$ cannot be $\perp$ to both CD and AB.	7. See § 93.

463. *The perpendicular segment between two skew straight lines is the shortest segment between them.*

PROPOSITION XXI. THEOREM

464. I. *Any point in the plane which is perpendicular to a segment at its mid-point is equidistant from the ends of the segment.*



Hypothesis. $AC = CB$. Plane $MN \perp AB$, through C.

Conclusion. Any point on MN is equidistant from A and B.

[Proof left to the pupil.]

II. *Any point which is equidistant from the ends of a segment is in the plane perpendicular to and bisecting the segment.*

Hypothesis. D is any point equidistant from A and B.

Conclusion. D lies in a plane perpendicular to AB at its mid-point.

Proof:

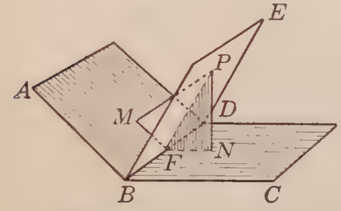
STATEMENTS	REASONS
1. D lies in a line $CD \perp$ to and bisecting AB.	1. § 116, II.
2. But CD lies in the plane $\perp AB$ at C.	2. § 422.
3. $\therefore D$ lies in the plane and bisecting AB.	3. Why?

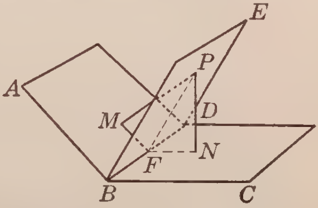
465. Cor. 1. *The locus of points in space equidistant from two points is the plane perpendicular to and bisecting the segment joining the points.*

466. Cor. 2. *The locus of points equidistant from the vertices of a triangle is the straight line perpendicular to the plane of the triangle, drawn through the center of the circumscribed circle of the triangle.*

PROPOSITION XXII. THEOREM

467. I. *Every point in the plane bisecting a diedral angle is equidistant from the faces of the angle.*





Hypothesis. Plane BE bisects diedral $\angle ABDC$.
P is any point in plane BE.
$PM \perp$ plane AD; $PN \perp$ plane DC.

Conclusion. $PM = PN$.

Plan. Prove PM and PN corres. parts of $\cong$ ≜.

Proof:

STATEMENTS	REASONS
1. PM and PN determine a plane which intersects AD, BE, and CD in FM, FP, and FN respectively.	1. Why?
2. Plane $PMFN \perp AD$, and also $\perp CD$.	2. Why?
3. $\therefore PMFN \perp BD$.	3. Why?
4. $\therefore$ ∠s PFM and PFN are the plane ∠s of diedral ∠s $ABDE$ and $CBDE$, respectively.	4. Prove it.

(Complete the proof.)

II. *Any point within a diedral angle and equidistant from its faces lies in the plane bisecting the diedral angle.*

Hypothesis. Planes AD and CD form diedral $\angle ABDC$.
$PM \perp$ plane AD; $PN \perp$ plane AC; $PM = PN$.

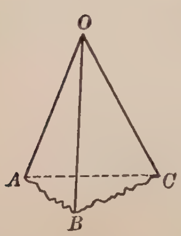
Conclusion. P lies in the plane bisecting $\angle ABDC$.

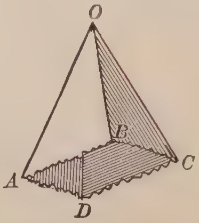
Plan. Prove plane PBD bisects the diedral angle.

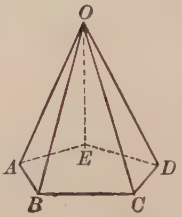
468. Cor. *The locus of points equidistant from two intersecting planes consists of the planes bisecting the diedral angles formed by the planes.*

POLYEDRAL ANGLES

469. The figures below represent **polyedral angles.**







Each is formed of portions of three or more intersecting planes, called the **faces** of the polyedral angle. The faces intersect in one common point, the **vertex** of the angle. Each face has two edges, the **edges** of the polyedral angle. On each face, the two edges form an angle, called a **face angle** of the polyedral angle. Each pair of consecutive faces forms a **diedral angle** of the polyedral angle.

Sometimes a plane is drawn, not through the vertex but intersecting all the faces, to aid in picturing the number of faces of the polyedral angle, as in $\angle O-ABCDE$.

A polyedral angle is **convex** if the polygon formed by a plane which intersects all its faces on one side of its vertex is a **convex polygon.** (§ 121.)

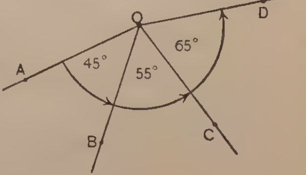
470. A **triedral angle** is a polyedral angle having three faces.

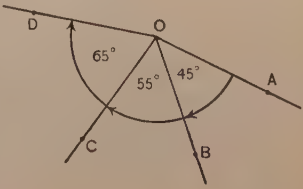
Ex. 69. (*a*) Construct from stiff paper two triedral angles, using the first pattern. Bend the paper along the lines *OB* and *OC* so that the letters are on the outside, and bring *OD* around to coincide with *OA*.

(*b*) Construct a third triedral angle, using the second pattern.

(*c*) Can you make your first two triedral angles coincide?

(*d*) Can you make your third triedral angle coincide with either of your first two?

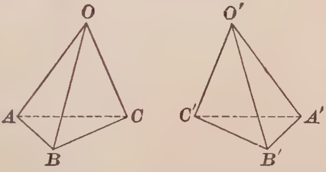




471. Two polyedral angles are **congruent** if they can be made to coincide. The parts of one are equal to the parts of the other and are arranged in the same order.

472. Two polyedral angles are **symmetric** if the face angles and the diedral angles of the one are equal respectively to the corresponding parts of the other, *but are arranged in opposite order.*

It is evident that *symmetric polyedral angles cannot be made to coincide.* (Ex. 69, *d.*)



473. Two polyedral angles are **vertical** if the edges of the one are the prolongations through the vertex of the edges of the other.

Obviously, vertical polyedral angles have the same vertex and the faces of the one lie on the same planes as the corresponding faces of the other. The face angles of the one equal the corresponding face angles of the other (why?), and the diedral angles of the one equal the corresponding diedral angles of the other. (See § 453.) But the parts of the one are arranged in opposite order to the corresponding parts of the other. Fig. 2 at the right is obtained from Fig. 1 by moving $\angle O - A'B'C'$ parallel to itself to the right, and then revolving it forward until face $OA'C'$ comes into the same plane as OAC. To pass from OA to OB, to OC, one moves from left to right; to pass from OA' to OB', to OC', one moves from right to left.

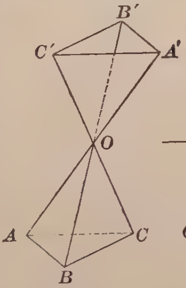


Fig. 1

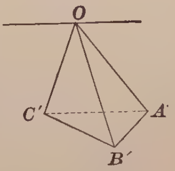


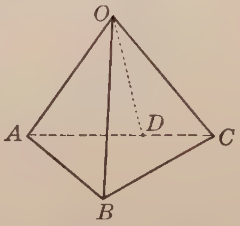
Fig. 2

474. As a consequence of the foregoing paragraphs, *two vertical polyedral angles are symmetric.*

PROPOSITION XXIII. THEOREM

475. *The sum of any two face angles of a triedral angle is greater than the third.*

Note. The theorem requires proof only when the third face angle is greater than either of the other two.



Hypothesis. In $\angle O\text{–}ABC$, $\angle AOC > \angle AOB$, and $\angle AOC > \angle BOC$.

Conclusion. $\angle AOC < \angle AOB + \angle BOC$.

Proof:

STATEMENTS	REASONS
1. In face AOC, draw OD making $\angle AOD = \angle AOB$.	1. $\angle AOC > \angle AOB$.
2. Draw AC, cutting AO at A, OD at D, and OC at C. Take $OB = OD$.	2. Possible constructions.
3. Plane ACB cuts the faces in AC, AB, and BC.	3. § 413.
4. $\triangle AOB \cong \triangle AOD$, and $AB = AD$.	4. Prove it.
5. In $\triangle ABC$, $AB + BC > AD + DC$.	5. Why?
6. $\therefore BC > DC$.	6. Ax. 10, § 148.
7. In $\triangle BOC$ and $\triangle COD$, $\angle BOC > \angle COD$.	7. Prove it, using § 156.
8. $\therefore \angle AOB + \angle BOC > \angle AOD + \angle COD$.	8. Ax. 9, § 148.
9. $\therefore \angle AOB + \angle BOC > \angle AOC$ or $\angle AOC < \angle AOB + \angle BOC$.	9. Why?

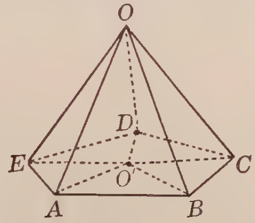
Ex. 70. Prove that any face angle of a polyedral angle is less than the sum of the remaining face angles.

Plan. Divide the polyedral $\angle$ into triedral $\measuredangle$ by passing planes through any lateral edge, and use § 475.

Ex. 71. What theorem of plane geometry is recalled by § 475?

PROPOSITION XXIV. THEOREM

476. *The sum of the face angles of any convex polyedral angle is less than four right angles.*



Hypothesis. $O\text{–}ABCDE$ is any convex polyedral $\angle$.

Conclusion. $\angle AOB + \angle BOC +$ etc. < 4 rt. ∠s.

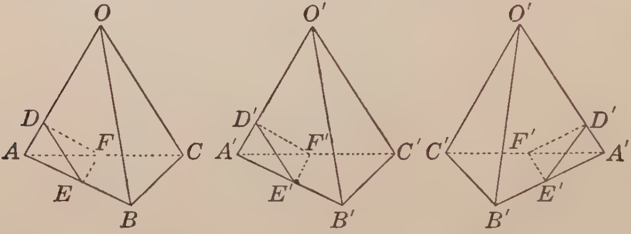
Proof:

	STATEMENTS	REASONS
1.	Pass a plane cutting the faces of the polyedral $\angle$ in the convex polygon $ABCDE$.	1. § 469.
2.	Join any point O' inside $ABCDE$ to A, B, C, D, and E, by segments $O'A$, $O'B$, $O'C$, $O'D$, and $O'E$.	2. § 5, *a*.
3.	In $\angle A\text{–}EOB$, $\angle OAE + \angle OAB > \angle EAB$; or $\angle OAE + \angle OAB > \angle EAO' + \angle O'AB$. Similarly, $\angle OBA + \angle OBC > \angle ABO' + \angle O'BC$; etc.	3. § 475.
4.	Adding, the sum of the base ∠s of the △s whose vertex is O is greater than the sum of the base ∠s of the △s whose vertex is O'.	4. Ax. 12, § 148.
5.	But, the sum of all the ∠s of the △s whose vertex is O equals the sum of all the ∠s of the △s whose vertex is O'.	5. Why?
6.	∴ the sum of the ∠s at O is less than the sum of the ∠s at O'.	6. Ax. 13, § 148.
7.	But the sum of the ∠s at $O' = 4$ rt. ∠s.	7. Why?
8.	∴ the sum of the ∠s at $O < 4$ rt. ∠s.	8. Why?

Note. Notice that the inequality in Step 3 does not mean that $\angle OAE > \angle O'AE$ and $\angle OAB > \angle O'AB$; rather, the sum of $\angle OAE$ and $\angle OAB$ is greater than the face angle EAB of triedral $\angle A\text{–}EOB$.

PROPOSITION XXV. THEOREM

477. *If two triedral angles have the face angles of the one equal respectively to the face angles of the other, their corresponding diedral angles are equal.*



Hypothesis. In triedral ∡ $O\text{–}ABC$ and $O'\text{–}A'B'C'$:
$\angle AOB = \angle A'O'B'$; $\angle BOC = \angle B'O'C'$; $\angle COA = \angle C'O'A'$

Conclusion. Diedral $\angle OA$ = diedral $\angle O'A'$; etc.

Proof:

STATEMENTS	REASONS
1. Make $OA = OB = OC = O'A' = O'B' = O'C'$; draw AB, BC, AC, $A'B'$, $B'C'$, and $A'C'$.	1. Possible.
2. $\triangle OAB \cong \triangle O'A'B'$, and $AB = A'B'$. Similarly $BC = B'C'$, and $AC = A'C'$.	2. Prove it.
3. $\angle BAC = \angle B'A'C'$.	3. Prove it.
4. On OA and $O'A'$, take $AD = A'D'$; draw $DE \perp OA$, cutting AB at E; draw $DF \perp OA$, cutting AC at F; draw EF. Similarly draw $D'E'$, $D'F'$, and $E'F'$.	4. All are possible constructions.
5. $\therefore \triangle ADE \cong \triangle A'D'E'$.	5. Prove it.
6. $\therefore AE = A'E'$, and $DE = D'E'$. Similarly $AF = A'F'$, and $DF = D'F'$.	6. Why?
7. $\therefore \triangle AEF \cong \triangle A'E'F'$, and $EF = E'F'$.	7. Prove it.
8. $\therefore \angle EDF = \angle E'D'F'$.	8. Prove it.
9. ∡ EDF and $E'D'F'$ are the plane ∡ of the ∡ OA and $O'A'$ respectively.	9. Why?
10. $\therefore$ diedral $\angle OA$ = diedral $\angle O'A'$. Similarly $\angle OB = \angle O'B'$; etc.	10. Why?

478. Cor. *If two triedral angles have the face angles of the one equal respectively to the face angles of the other:*

(a) they are congruent if the equal parts are arranged in the same orders;

(b) they are symmetric if the equal parts are arranged in opposite orders.

Observe that the proof of § 477 applies for Fig. 3, as well as for Fig. 2. This corollary therefore is an immediate consequence of § 477, § 471, and § 472.

Note. Further theorems about triedral angles appear as Optional Topic C, on page 351.

Review Exercises

Ex. 72. When do two given straight lines determine a plane?

Ex. 73. Do three points always determine a plane?

Ex. 74. Is a trapezoid a plane figure?

Ex. 75. How many planes can be passed parallel to a given plane:
(*a*) through a given point not in the given plane?
(*b*) through a given line parallel to the given plane?
(*c*) through a given line not parallel to the given plane?

Ex. 76. How many planes can be passed perpendicular to a given plane:
(*a*) through a given point?
(*b*) through a given line not perpendicular to the given plane?
(*c*) through a given line perpendicular to the given plane?

Ex. 77. Are lines in space parallel if they are:
(*a*) perpendicular to the same straight line?
(*b*) if they are perpendicular to the same plane?

Ex. 78. How many lines can be drawn ⊥ to the same plane?

Ex. 79. Are planes parallel if they are:
(*a*) perpendicular to the same straight line?
(*b*) if they are perpendicular to the same plane?

Ex. 80. Is a line perpendicular to a plane if it is perpendicular to a line of the plane?

Ex. 81. Can two intersecting planes be perpendicular to:
(*a*) the same straight line?
(*b*) to the same plane?

Ex. 82. (*a*) If a line is parallel to a plane, is it parallel to every line in the plane?

(*b*) Is it parallel to any line or lines in the plane?

Ex. 83. In the figure for § 443, if $AB = BC$ and $A'C' = 20''$, how long is $A'B'$?

What can you infer in each of the exercises 84 to 100 inclusive, if the facts stated in the exercise are known:

(Draw a figure, *if necessary*, to represent the given conditions.)

Ex. 84. Line $XY \perp AB$ at A and $XY \perp AC$ at A.

Ex. 85. Line $XY \parallel$ line ZW and $XY \perp$ plane MN.

Ex. 86. AX, AY, and AZ, are all $\perp$ to line RS at A.

Ex. 87. Line $AB \parallel$ line CD and plane MN contains CD.

Ex. 88. Line $XY \parallel$ line AB and line $RS \parallel AB$.

Ex. 89. Line $XY \perp$ plane MN, and line $RS \perp$ plane MN.

Ex. 90. Line $RS \parallel$ plane MN, and plane XY, containing RS, intersects MN in line CD.

Ex. 91. Plane $MN \parallel$ plane RS, and line $AB \perp$ plane MN.

Ex. 92. Plane $MN \parallel$ plane XY, and plane RS cuts MN in AB and XY in CD.

Ex. 93. Lines AB and CD intersect and form plane MN. $AB \parallel$ plane RS and $CD \parallel$ plane RS.

Ex. 94. Plane $MN \perp$ line AB, and plane $RS \perp$ line AB.

Ex. 95. Lines AB and XY are parallel, also line $AC \parallel$ line XY.

Ex. 96. Plane $MN \perp$ plane RS. A is on their intersection. $AB \perp RS$.

Ex. 97. Line $AB \perp$ plane MN. Plane CD contains AB.

Ex. 98. Plane $MN \perp$ plane RS. Line AB is in MN. $AB \perp$ the intersection of MN and RS.

Ex. 99. Plane $MN \perp$ plane RS. A is in MN, and line $AB \perp RS$.

Ex. 100. Plane MN intersects plane RS in line AC. Plane $XY \perp MN$, and $XY \perp RS$.

Ex. 101. Can a triedral angle have face angles which measure 130°, 40°, and 80°?

Ex. 102. P is the mid-point of line AB. Plane MN contains P and $MN \perp AB$. R is on MN, and is $8''$ from A. How far is R from B?

Ex. 103. A point is 3 ft. from the front wall of a room and also 3 feet from one of the side walls. In what plane must it lie?

Ex. 104. If a line in one face of a diedral angle is parallel to one in the other face, prove that each is parallel to the edge of the angle.

Ex. 105. If parallel planes cut off equal distances on one transversal, they cut off equal distances on all transversals.

Ex. 106. A line is parallel to one of two perpendicular planes and perpendicular to the other. Prove that a plane containing this line and intersecting the first given plane intersects that plane in a line which is perpendicular to the intersection of the two given planes.

Ex. 107. If a line and a plane are parallel:

(*a*) is a plane which is perpendicular to the line necessarily perpendicular to the given plane?

(*b*) is a plane which is perpendicular to the given plane necessarily perpendicular to the given line?

Ex. 108. Prove that a plane can be passed through a given point so that it will be perpendicular to each of two given planes.

Ex. 109. If each of three planes is perpendicular to each of the other two, then each line of intersection is perpendicular to each of the other intersections.

Ex. 110. A carpenter's level is used to determine whether a plane is level. In how many directions must a plane be tested in order to assure that it is level?

Ex. 111. If a line is tangent to a circle, it is perpendicular to the line drawn from the point of tangency to any point on the perpendicular to the plane of the circle, erected at the center of the circle. (Use Ex. 34.)

Ex. 112. What is the locus of points equidistant from two points A and B and also equidistant from two points C and D?

Ex. 113. What is the locus of points on the floor of a room equidistant from the front and one of the side walls of the room?

Ex. 114. What is the locus of points in plane CD equidistant from two planes MN and RS which form an angle of 40°?

Ex. 115. Prove that the segment joining the midpoints of one pair of opposite sides of a skew quadrilateral bisects the segment joining the midpoints of the other pair.

Ex. 116. Prove that two angles not in the same plane are equal if their sides are parallel each to each, if the parallels extend in opposite directions from their respective vertices.

Note. Additional Exercises 12–15, page 458, can be done now.

OPTIONAL TOPICS

Topic A. **Loci in Space.**
Topic B. **Projections of Figures.**
Topic C. **Analogy between Triangles and Triedral Angles.**

OPTIONAL TOPIC A

479. *The locus of points in space at a given distance from a given plane consists of two planes, parallel to the given plane and at the given distance from it.* (Easily proved.)

480. *The locus of points in space equidistant from two parallel planes is the plane parallel to them and mid-way between them.* (This also is easily proved.)

481. Intersection of loci. When two or more conditions are imposed upon a point in space, each condition determines a locus for that point, and the desired point lies in the intersection of the loci.

The following form of solution brings out all the value of such a problem even better than if a figure were drawn.

Illustrative Problem. What is the locus of points in space at a given distance from a given plane and equidistant from two given points?

Solution. 1. The locus of points at a given distance from a given plane consists of two planes parallel to the given plane and at the given distance from it. Call this Locus 1.

2. The locus of points equidistant from two given points is the plane perpendicular to and bisecting the segment between the two points. Call this Locus 2.

3. The desired locus is the intersection of Locus 1 and Locus 2.

Discussion. 1. Generally the plane of Locus 2 will intersect the planes of Locus 1 in two straight lines.

2. Locus 2 may be parallel to the planes of Locus 1. In this case, there will not be any points satisfying the given conditions.

3. Locus 2 may coincide with one of the planes of Locus 1. The plane common to Loci 1 and 2 will be the desired locus.

Ex. 117. In a given plane, what is the locus of points equidistant from A and B when segment AB is 12 in. long?

Note. A and B should be assumed not in the given plane.

Ex. 118. In a given plane, what is the locus of points 6 in. from a second given plane?

Ex. 119. In a given plane, what is the locus of points equidistant from two parallel planes which are 10 in. apart?

Ex. 120. In a given plane, what is the locus of points equidistant from the faces of a given diedral angle?

Ex. 121. In a given plane, what is the locus of points equidistant from two intersecting planes?

Ex. 122. Find a point which is equidistant from the vertices of a given triangle and also equidistant from two parallel planes.

Ex. 123. What is the locus of points equidistant from the front wall and the two side walls of a room?

Ex. 124. Let AA', BB', and CC' lie above the plane of $\triangle ABC$ and be perpendicular to it at A, B, and C, respectively. What is the locus of points equidistant from the planes AB', BC', and AC'?

Ex. 125. What is the locus of points equidistant from two given points and also equidistant from two parallel planes?

Ex. 126. What is the locus of points equidistant from two parallel planes, and also equidistant from two intersecting planes?

Ex. 127. What is the locus of points equidistant from two given points and also equidistant from two intersecting planes?

Ex. 128. What is the locus of points at a given distance from a given plane, and also equidistant from two given points?

Ex. 129. What is the locus of points equidistant from two intersecting planes and also at a given distance from a given plane?

Ex. 130. At most, how many points are there which are at a given distance from a given plane, equidistant from two other parallel planes, and equidistant from two given points?

Ex. 131. At most, how many points are there which are equidistant from four given points, of which three are the vertices of a triangle, and the fourth lies outside the plane of the triangle?

Ex. 132. At most, how many points are there which are equidistant from two intersecting planes, equidistant from two other parallel planes, and also equidistant from two given points?

Ex. 133. Are there any points which are equidistant from the front wall, the two side walls, and the ceiling of a room?

Ex. 134. What is the locus of points on the floor of a room which is 10 ft. high, if the points are to be 12 ft. from a point on the ceiling of the room?

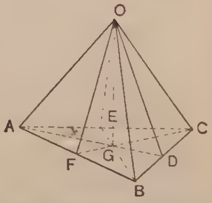
Ex. 135. What is the locus of points which are 5 in. from plane *MN* and also 10 in. from plane *RS*?

Ex. 136. In a given line, find all points which are equidistant from two intersecting planes.

Ex. 137. At most, how many points are there which are 8 in. from one given plane, 10 in. from a second given plane, and 12 in. from a third given plane?

Ex. 138. Prove that the planes bisecting the diedral angles of a triedral angle meet in a line, whose points are equidistant from the faces of the triedral angle.

Plan. Planes *OAD* and *OBE* intersect in line *OG*. Prove *OG* lies in plane *OCF*.



Ex. 139. Prove that, at most, there is one point which is equidistant from the three faces of a triedral angle, and at the same time equidistant from two given points.

Ex. 140. Prove that the three planes determined by the edges of a triedral angle and the bisectors of the opposite face angles intersect in one straight line.

Plan. 1. In a figure like that for Ex. 138, assume that *OD*, *OF*, and *OE* are the bisectors of $\angle$s *BOC*, *BOA*, and *AOC*, respectively.
2. Take $OA = OB = OC$, and draw *AB*, *AC*, and *BC*.
3. Prove *AD*, *CF*, and *EB* concurrent.
4. Prove planes *AOD*, *BOE*, and *COF* meet in a line.

Ex. 141. Prove that the locus of points equidistant from the sides of an angle is the plane through the bisector of the angle and perpendicular to the plane of the angle.

Suggestions. Recall § 237 and Ex. 65, Book VI, page 333.

Ex. 142. Prove that the planes through the bisectors of the face angles of a triedral angle, perpendicular to the faces of the triedral angle, meet in a line, whose points are equidistant from the edges of the triedral angle.

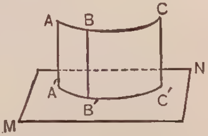
Ex. 143. At most how many points are there inside a triedral angle which are equidistant from the edges of a triedral angle and 5 in. from the vertex of the angle?

Optional Topic B. Projections on a Plane

482. The **projection of a point on a plane** is the foot of the perpendicular to the plane from the point.

The **projection of a line upon a plane** consists of the projections on the plane of all points of the line.

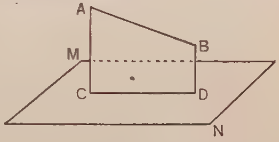
Thus, $A'B'C'$ is the projection of ABC on MN.



483. *The projection on a plane of a straight line which is not perpendicular to the plane is a straight line.*

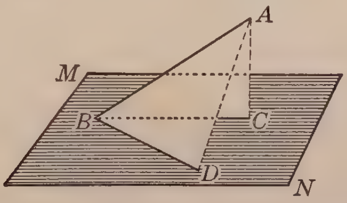
Plan. 1. Through AB, pass a plane $AD \perp MN$.

2. Prove that the feet of the ⊥s to MN from AB lie in CD. (§ 458.)



PROPOSITION XXVI. THEOREM

484. *The acute angle which a straight line makes with its projection on a plane is the least angle which it makes with any line in the plane.*



Hypothesis. BC is the projection of BA on plane MN.
BD is any other line in MN through B.

Conclusion. $\angle CBA < \angle DBA$.

Plan. 1. Take $BD = BC$. 2. Compare AD and AC.
3. Compare $\angle CBA$ and DBA, using § 156.

485. The **inclination of a line to a plane** is the acute angle made by the line and its projection on the plane.

486. *The projection on a plane of a segment equals the length of the segment multiplied by the cosine of the inclination of the segment to the plane.* $\left(\text{For } \frac{BC}{AB} = \cos \angle CBA.\right)$

Ex. 144. Can the projection on a plane of a curved or broken line be a straight line?

Ex. 145. If the projection on a plane of a given figure is a straight line, then the figure lies in a plane.

Ex. 146. Let the projection on a given plane *MN* of a segment of length l be denoted by l'. What is the relation between l and l' if: (*a*) $l \perp$ plane *MN*? (*b*) $l \parallel$ plane *MN*? (*c*) l and *MN* make an angle of 45°?

Ex. 147. (*a*) A segment 10 in. long is inclined to a plane at an angle of 60°. How long is its projection on the plane? (*b*) Repeat part (*a*) if the angle of inclination is 30°. (*c*) Repeat part (*a*) if the angle of inclination is 45°.

Ex. 148. (*a*) A segment 25 in. long is inclined to a plane at an angle of 40°. How long is its projection on the plane?

Suggestion. Use § 486. To find cos 40°, use the table, p. 500. (*b*) Repeat part (*a*) when the angle is 75°.

Ex. 149. If two equal segments are drawn to a plane from a point not in the plane, they make equal angles with the plane.

Ex. 150. If two parallels meet a plane, they make equal angles with the plane.

Ex. 151. If two parallel segments are oblique to a plane, they have the same ratio as their projections on the plane.

Ex. 152. If two parallels are oblique to a plane, their projections on the plane are parallel.

Ex. 153. If a straight line intersects two parallel planes, it makes equal angles with the planes.

Ex. 154. If a straight line is perpendicular to one of two perpendicular planes, its projection on the other is perpendicular to the intersection of the two planes.

Ex. 155. A rectangle *ABCD* is inclined to a plane *MN* in such manner that sides *AB* and *CD* are parallel to *MN*, but *BC* and *AD* have an inclination of 60° to *MN*. If *AB* is 10 in. long and *BC* is 20 in. long, what kind of figure is the projection of *ABCD* on *MN*?

Suggestion. It can be proved that $AB \perp$ plane $BB'CC'$.

Ex. 156. Triangle *ABC* is so inclined to plane *MN* that *AB* is parallel to *MN* and the altitude *CD* of $\triangle ABC$ makes an angle of 30° with *MN*. If *AB* is 18 in. and *CD* is 12 in.:

(*a*) find the area of $\triangle ABC$.

(*b*) find the area of the projection of $\triangle ABC$ on plane *MN*.

Note. Additional Exercises 16 to 22, p. 458, can be done now.

OPTIONAL TOPIC C

THE ANALOGY BETWEEN TRIANGLES AND TRIEDRAL ANGLES

Many propositions about triangles can be changed into propositions about triedral angles by substituting for the word "angle" of the former the words *diedral angle*, and for the word "side" of the former the words *face angle*.

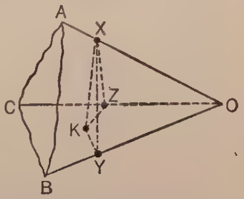
Ex. 157. Two triedral angles are congruent if a face angle and the adjacent diedral angles of the one are equal respectively to a face angle and the adjacent diedral angles of the other, provided the parts are arranged in the same orders.

Note. If the parts of one are arranged in the opposite order to the parts of the other, the $\measuredangle$ cannot be made to coincide. In this case, form the vertical triedral angle to one of the given triedrals. It is of course symmetric to that given triedral. Its parts therefore will be equal to the parts of the other given triedral and will be arranged in the same order as are those of that triedral. It therefore is congruent to the second given triedral. Therefore the second given triedral must be symmetric to the first given triedral.

Ex. 158. Two triedral angles are congruent if two face angles and the included diedral angle of the one are equal respectively to two face angles and the included diedral angle of the other, provided they are arranged in the same order. They are symmetric if the equal parts are arranged in opposite orders.

Ex. 159. If two face angles of a triedral angle are equal, the opposite diedral angles also are equal. (Recall the proof of § 68.)

Ex. 160. If two diedral angles of a triedral $\angle$ are equal, the opposite face angles are equal.



Suggestion. **1.** Draw $XK \perp$ plane BOC; $KY \perp BO$; $KZ \perp CO$; draw XY and XZ.

2. Recall Ex. 34, p. 321.

Note. It is assumed in this figure that K falls inside $\angle BOC$. Other possible positions of point K might be considered.

Ex. 161. State and prove the theorem corresponding to: *the bisector of the vertex angle of an isosceles triangle is perpendicular to and bisects the base.*

Ex. 162. If two diedral angles of a triangle are unequal, the opposite face angles are unequal in the same order.

Note. Additional Exercise 24, p. 459, can be done now.

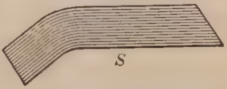
BOOK VII

POLYEDRONS

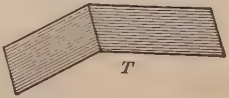
487. Surfaces other than plane surfaces.

A **curved surface** is a surface of which no part is plane.

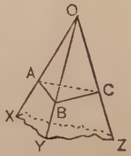
There are surfaces of which part is plane and part is curved; as surface S.

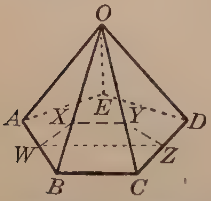


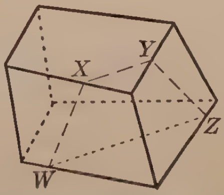
There are surfaces consisting of two or more parts, each of which is plane; as surface T.

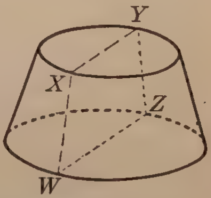


488. Closed surfaces. Let the plane ABC intersect the faces of triedral angle $O\text{–}XYZ$. The surface consisting of the triangles OAB, OAC, OBC, and ABC, and the parts of their planes inclosed by them is a closed surface. Below are other examples of closed surfaces. *Every plane intersects such surfaces in a closed line;* as $XYZW$.









489. A closed surface is convex if the intersection of it and every intersecting plane is a *convex closed line.*

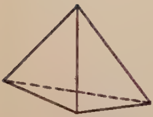
A convex closed line is such that a straight line will cut it in only two points.

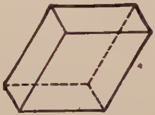
490. *A convex closed surface incloses a finite portion of space,* which is called a **solid.** The surface is the **surface of the solid,** and is said to bound the solid. The space inclosed by the surface is the **interior of the solid.**

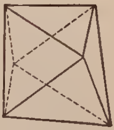
491. A **polyedron** is a solid bounded by portions of planes, called the **faces** of the polyedron. The faces intersect in straight lines, called the **edges** of the polyedron. The edges intersect in points, called the **vertices** of the polyedron. The straight line joining any two vertices of the polyedron which do not lie in the same face is a **diagonal** of the polyedron.

492. The least number of planes that can form a polyedral angle is three. Then the least number of planes that can form a polyedron is four.

A polyedron having four faces is a **tetraedron**; one having six faces is a **hexaedron**; one having eight faces is an **octaedron**; one having twelve faces is a **dodecaedron**; and one having twenty faces is an **icosaedron.**


TETRAEDRON

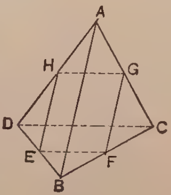

HEXAEDRON


OCTAEDRON

Ex. 1. Verify for a tetraedron, an octaedron, and a hexaedron, the formula

$$F + V = E + 2,$$

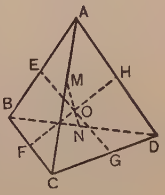
where F is the number of faces, V is the number of vertices, and E is the number of edges.



Ex. 2. If E, F, G, and H are the mid-points of edges BD, BC, AC, and AD, respectively, of tetraedron $ABCD$, prove $EFGH$ a parallelogram.

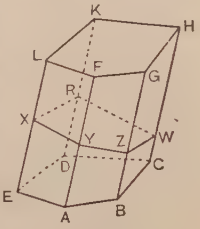
Ex. 3. If $ABCD$ is a tetraedron, the section made by a plane parallel to each of the edges AB and CD is inclosed by a parallelogram.

Note. Remember that by § 432 a plane can be drawn parallel to each of two skew straight lines.



Ex. 4. The lines joining, by pairs, the mid-points of opposite edges of any tetraedron, intersect in a common point.

493. A **prism** is a polyhedron, two of whose faces lie in parallel planes, and whose remaining faces, in order, intersect in parallel lines. The parallel faces are the **bases**; the other faces are the **lateral faces**; the edges which are not sides of the bases are the **lateral edges**; the perpendicular between the planes of the bases is the **altitude**; the sum of the areas of the lateral faces is the **lateral area.**



If a plane cutting all the lateral edges of a prism is perpendicular to one of them, its intersection with the prism is called a **right section of the prism**, as $XYZWR$. The plane is obviously perpendicular to all the edges.

494. The following **important facts about a prism** should be proved by the pupil (Fig. § 493):

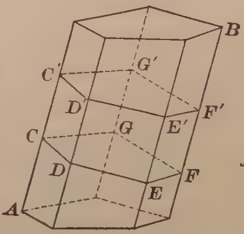
I. *The lateral faces of a prism are inclosed by parallelograms.*

II. *The lateral edges of a prism are parallel and equal.*

III. *The bases of a prism are inclosed by congruent polygons.*

***IV.** *Sections of the lateral surface of a prism made by two parallel planes cutting all the lateral edges are congruent polygons.*

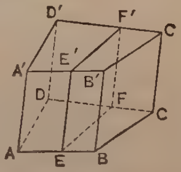
Suggestion. Prove $CDEFG \cong C'D'E'F'G'$.



V. *A section of a prism made by a plane that is parallel to a base is congruent to that base.*

Ex. 5. Prove that every pair of lateral edges of a prism determines a plane which is parallel to each of the other lateral edges of the prism.

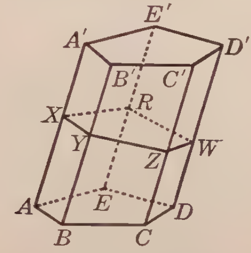
Ex. 6. Prove that the section of a prism made by a plane parallel to a lateral edge is inclosed by a parallelogram.



Suggestion. Let plane EF' be ∥ AA', a lateral edge of prism AC'.

*PROPOSITION I. THEOREM

495. *The lateral area of a prism equals the perimeter of a right section multiplied by the length of a lateral edge.*



Hypothesis. $XYZWR$ is a rt. section of prism AC'.
p = perimeter of $XYZWR$; e = length of AA';
S = the lateral area of AC'.

Conclusion. $S = ep$

Plan. Prove area $\square AB' = e \times XY$; etc. Then add.

496. Kinds of prisms. A prism is *triangular, quadrangular,* etc., according as its base is *triangular, quadrangular,* etc.

A **right prism** is a prism one of whose lateral edges is perpendicular to its bases; hence, all the lateral edges are perpendicular to the bases.

An **oblique prism** is a prism whose lateral edges are not perpendicular to its bases.

497. Cor. *The lateral area of a right prism equals the perimeter of the base multiplied by the length of the altitude.*

Ex. 7. The lateral edges of a right prism are equal to the altitude.

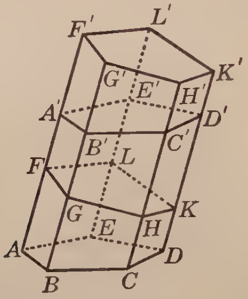
Ex. 8. The lateral faces of a right prism are inclosed by rectangles.

498. The **volume of a solid** is the number of units of solid measure which it contains.

499. Two solids are equal if they have equal volumes. *Two congruent solids are equal;* also *two solids which consist of parts which are respectively congruent are equal.*

* PROPOSITION II. THEOREM

500. *An oblique prism equals a right prism whose base is a right section of the oblique prism, and whose altitude is a lateral edge of the oblique prism.*



Hypothesis. FK' is a right prism. Its base FK is a right section of oblique prism AD'. Its altitude $FF' = AA'$, a lateral edge of AD'.

Conclusion. Prism FK' = prism AD'.

Proof:

	Statements	Reasons
1.	$ABCDE \cong A'B'C'D'E'$; $FGHKL \cong F'G'H'K'L'$.	1. Why?
2.	$AF = A'F'$; $BG = B'G'$; etc.	2. Prove it.
3.	Slide AK so that AF, BG, etc., move along AF', BG', etc., until A falls on A'.	3. § 60.
4.	∴ AB falls on $A'B'$, and B falls on B'. Similarly C, D, and E fall on C', D', and E'.	4. Why?
5.	F, G, H, etc., will fall on F', G', H', etc.	5. Why?
6.	∴ polyedron $AK \cong$ polyedron $A'K'$.	6. Why?
7.	∴ prism FK' = prism AD'.	7. § 499.

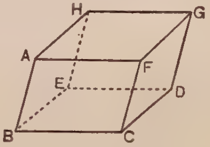
501. I. *Two right prisms are congruent, and hence equal, when they have congruent bases and equal altitudes.*

II. *Two prisms are congruent if three faces which include a triedral angle of one are congruent respectively to three faces which include a triedral angle of the other, and are similarly placed.* (Prove it by superposition.)

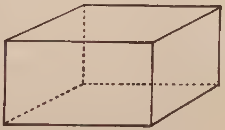
Ex. 9. There are upon a porch six columns having the form of right prisms with regular octagonal bases. If the side of the base of each is 6 in. and the altitude of each is 8 ft., find the sum of the lateral areas of the columns.

Ex. 10. What is the lateral area of the right prism whose altitude is 3 ft. 6 in. and whose base is inclosed by an equilateral triangle whose side is 1 ft.?

Ex. 11. What is the total area of the right prism whose base is inclosed by a regular hexagon whose side is 5 in. long, if the altitude of the prism is 25 in. long?



502. A **parallelepiped** is a prism whose base is inclosed by a parallelogram. Hence all the faces are inclosed by parallelograms.



A **right parallelepiped** is a parallelepiped in which one lateral edge and, hence, *all the lateral edges are perpendicular to the bases.*

A **rectangular parallelepiped** (or a rectangular solid) is a right parallelepiped whose base is inclosed by a rectangle. As a consequence *all the faces are inclosed by rectangles.* (See Ex. 8.)

A **cube** is a rectangular parallelepiped whose six faces are inclosed by squares.

Ex. 12. Are any faces of a right parallelepiped rectangular?

Ex. 13. Is a cube a prism?

Ex. 14. What is the total area of the cube whose edge is 6 in.?

Ex. 15. Repeat Ex. 14, if the edge is of length e.

Ex. 16. How long is the diagonal of the cube of edge 5 in.?

Ex. 17. Repeat Ex. 16, if the edge is of length e.

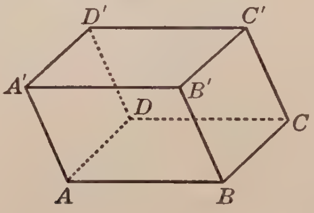
Ex. 18. How long is each diagonal of the rectangular parallelepiped whose edges are 6 in., 8 in., and 10 in. long?

Ex. 19. What is the lateral area of a prism whose lateral edge is 18 in., and whose right section is a triangle whose sides are 5 in., 7 in., and 8 in.?

Ex. 20. The polyedron formed by three pairs of parallel planes is a parallelepiped. (Use the first figure of § 502.)

PROPOSITION III. THEOREM

503. *The opposite lateral faces of a parallelepiped are congruent and parallel.*



Hypothesis. AC and $A'C'$ are the bases of parallelepiped AC'.

Conclusion. I. $AB' \parallel DC'$. II. $AB' \cong DC'$.

Plan. I. Use § 442. II. Prove AB' and DC' mutually equiangular and mutually equilateral.

504. Cor. *Any pair of opposite faces of a parallelepiped may be taken as its bases.* (Use § 493, and § 502.)

Ex. 21. A parallelepiped is rectangular if the planes meeting at one vertex are mutually perpendicular.

Ex. 22. Prove that the diagonals of a rectangular parallelepiped are equal.

Ex. 23. Prove that the diagonals of a parallelepiped bisect each other.

Note. The point of intersection of the diagonals of a parallelepiped is called the center of the parallelepiped.

Ex. 24. Prove that any line-segment drawn through the center of a parallelepiped, terminating in a pair of opposite faces, is bisected at that point.

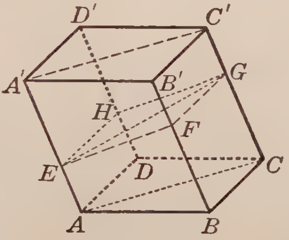
Ex. 25. Prove that the line-segment joining the center of a parallelepiped to the center of any face is parallel to any edge of the parallelepiped which intersects that face, and is equal to one half of it.

Ex. 26. Prove that the centers of two opposite faces of a parallelepiped and the center of the parallelepiped are collinear.

Suggestion. Recall Ex. 23 and § 89.

* PROPOSITION IV. THEOREM

505. *The plane through two diagonally opposite edges of a parallelepiped divides it into two equal triangular prisms.*



Hypothesis. Plane AC' passes through edges AA' and CC' of parallelepiped $A'C$.

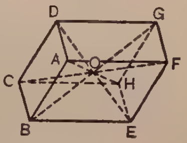
Conclusion. Prism $A' - ABC$ = prism $A' - ACD$.

Plan. Prove $A' - ABC$ and $A' - ACD$ equal, respectively, to equal right prisms.

Proof:

STATEMENTS	REASONS
1. Let $EFGH$ be a right section of the parallelepiped, intersecting plane AC' in EG.	1. § 493.
2. $EFGH$ is a ▱.	2. Prove it.
3. $\therefore \triangle EFG \cong \triangle EHG$.	3. Why?
4. Prism A'–ABC = a right prism with base EFG and altitude AA'. Prism A'–ACD = a right prism with base EHG and altitude AA'.	4. § 500.
5. But these right prisms are equal.	5. § 501.
6. $\therefore A'$–$ABC = A'$–ACD	6. Why?

Note. Prisms $A' - ABC$ and $A' - ACD$ are not congruent.



Ex. 27. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelepiped.

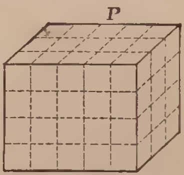
Plan. Prove $BCHE$ is a ▱ by proving BC = and ∥ HE.

VOLUME OF A PARALLELEPIPED

506. The **dimensions of a rectangular parallelepiped** are the lengths of the three edges which meet at one vertex.

These lengths (like areas and volumes) are numbers; they are the measures of the edges in terms of a common unit of linear measure. But the edges may be incommensurable (§ 205); and even more so, they may not contain an arbitrary unit of measure (like one inch) an integral number of times. This fact introduces difficulty in finding the volume of a rectangular parallelepiped.

If the dimensions of a rectangular parallelepiped are 5, 3, and 4, then, obviously, the solid contains 5 × 3 × 4 cubes of edge 1. Hence its volume is 60.



If the dimensions were 5, 3, and $4\frac{1}{3}$, then the solid would contain 60 such cubes and, besides, a layer of 15 pieces, each having dimensions 1, 1, and $\frac{1}{3}$. These 15 pieces could be built up into 5 cubes. Hence the volume would be 65. This number 65 can be obtained by finding $5 \times 3 \times 4\frac{1}{3}$.

If the dimensions were 5, 3, and 4.1, the volume would be a little more than 5 × 3 × 4. The foregoing examples suggest that we accept 5 × 3 × 4.1 as the volume.

507. Definition. *The volume of a rectangular parallelepiped is the product of its three dimensions,* when the edges are measured by the same unit of linear measure.

508. Cor. 1. *The volume of a rectangular parallelepiped equals the product of its base and the altitude to it.*

§ 508 is expressed in its commonly abbreviated form. Expressed more accurately, it would be: *the volume of a rectangular parallelepiped equals the product of the area of its base and the length of the altitude to it.*

The customary brief form of statement will be employed in the remaining theorems relating to area and volumes.

509. Cor. 2. *Two rectangular parallelepipeds have the same ratio as the products of their three dimensions.*

If the dimensions of P are a, b, and c, and of P' are a', b', and c', then, since the volume of $P = abc$, and the volume of $P' = a'b'c'$, then

$$\frac{P}{P'} = \frac{abc}{a'b'c'}.$$

510. Cor. 3. *Two rectangular parallelepipeds having equal bases are to each other as their altitudes.*

If parallelepipeds P and P' have bases B and B', and altitudes h and h', and if $B = B'$,

$$\text{then } \frac{P}{P'} = \frac{Bh}{B'h'} \text{ reduces to } \frac{P}{P'} = \frac{h}{h'}.$$

511. Cor. 4. *Two rectangular parallelepipeds having equal altitudes are to each other as their bases.*

If $h = h'$, then

$$\frac{P}{P'} = \frac{Bh}{B'h'} \text{ reduces to } \frac{P}{P'} = \frac{B}{B'}.$$

Ex. 28. Find the ratio of the volumes of two rectangular parallelepipeds whose dimensions are 8, 12, and 21, and 16, 15, and 36, respectively.

Ex. 29. Find the altitude of a rectangular parallelepiped, the dimensions of whose base are 21 and 60, equal to a rectangular parallelepiped whose dimensions are 42, 36, and 35.

Ex. 30. What must be the height of a tank having the form of a rectangular parallelepiped whose base has the dimensions 3 ft. and 10 ft., in order that the tank will contain 675 gal. of water when the water rises to within one foot of the top? (One cu. ft. of water $= 7\frac{1}{2}$ gal. approximately.)

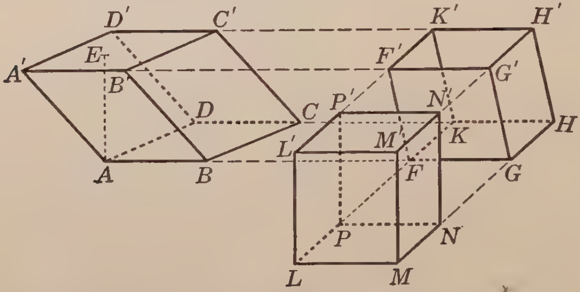
Ex. 31. Approximately how many barrels of water will run into a cistern during a $\frac{1}{2}$ in. fall of rain from the roof of a barn whose total roof area is 900 sq. ft.

Ex. 32. (*a*) How many cubic yards of concrete are required for the foundation walls of a house 25 ft. × 35 ft., if the walls are 9 in. thick and are 8 ft. high? (*b*) How many bags of cement are required if the mixture contains 4 bags of cement to one cubic yard of concrete?

Note. Exercises 24–30, page 459, can be studied now.

PROPOSITION V. THEOREM

512. *The volume of any parallelepiped is equal to the product of its base and altitude.*



Hypothesis. AC' is an oblique parallelepiped.

Let the length of altitude AE be h, the area of base $ABCD$ be B, and the volume of AC' be V.

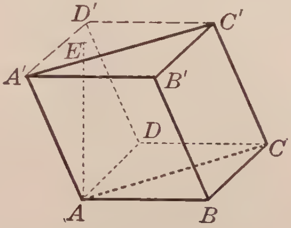
Conclusion. $V = Bh$.

Proof:

	Statements	Reasons
1.	Extend AB, DC, $A'B'$, and $D'C'$. Take $FG = AB$.	1. Possible.
2.	Pass planes FK' and $GH' \perp FG$, forming right parallelepiped FH'. (FK' and GH' are the bases.)	2. § 502.
3.	Prism FH' = prism AC'.	3. § 500.
4.	Extend HG, $H'G'$, $K'F'$, and KF. Take $NM = HG$	4. Possible.
5.	Pass planes NP' and $ML' \perp NM$, forming rectangular parallelepiped LN'.	5. See Ex. 20, and Ex. 21.
6.	Prism LN' = prism FH'.	6. Why?
7.	$\therefore$ Prism LN' = prism AC'.	7. Why?
8.	Volume LN' = base $LN \times MM'$	8. Why?
9.	But area LN = area $ABCD$, or B.	9. Prove it.
10.	Also $MM' = h$.	10. Why?
11.	$\therefore$ Volume $AC' = Bh$.	11. Ax. 2, § 49.

* PROPOSITION VI. THEOREM

513. *The volume of a triangular prism equals the product of its base and altitude.*



Hypothesis. $A'\text{–}ABC$ is a triangular prism.

Let h = the length of altitude AE; B = the area of base ABC; and V = the volume of $A'\text{–}ABC$.

Conclusion. $V = Bh.$

Plan. From $A'\text{–}ABC$, form parallelepiped $D'B$. Express the volume of $D'B$. Use § 505.

Note. The pupil should memorize now the following formulas:

1. Area of a △ $= \sqrt{s(s-a)(s-b)(s-c)}$, § 332, where the sides are a, b, and c, and $s = \frac{1}{2}(a+b+c)$.

2. Area of an equilateral △ of side $a = \frac{a^2}{4}\sqrt{3}$. (Ex. 39, p. 211.)

3. Altitude of an equilateral △ of side $a = \frac{a\sqrt{3}}{2}$.

Ex. 33. Find the volume of the right prism whose altitude is 15, and whose base is inclosed by an equilateral triangle with side 6.

Ex. 34. Find the volume of the right prism whose altitude is h, and whose base is inclosed by an equilateral triangle with side a.

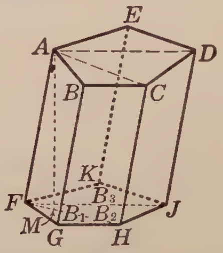
Ex. 35. Find the lateral area and volume of a right triangular prism, having the sides of its base 10, 8, and 14, respectively, and the altitude 20.

Ex. 36. Prove that the volume of a right triangular prism is equal to the product of the area of any face and one half the altitude to that face.

Ex. 37. The volume of a right triangular prism is $96\sqrt{3}$. Its base is inclosed by an equilateral triangle with side 8. Find its lateral area.

PROPOSITION VII. THEOREM

514. *The volume of any prism equals the product of its base and altitude.*



Hypothesis. Let h = the length of altitude AM, B = the area of base $FGHJK$, and V = the volume of prism AJ.

Conclusion. $V = Bh$.

Plan. 1. Through edge AF and diagonals FH and FJ of the base, pass planes $AFHC$ and $AFJD$, dividing prism P into triangular prisms P_1, P_2, and P_3, whose base areas are B_1, B_2, and B_3, respectively, and whose common altitude is of length h.

2. Express the volumes of P_1, P_2, and P_3. Add the results and simplify, thus obtaining an expression for the volume P.

515. Cor. 1. *Two prisms having equal bases and equal altitudes are equal.*

Suggestion. Let prisms P and P' have equal bases B and B', respectively, and equal altitudes h and h'. Prove $P = P'$. See page 361.

516. Cor. 2. *Two prisms having equal altitudes have the same ratio as their bases.*

517. Cor. 3. *Two prisms having equal bases have the same ratio as their altitudes.*

518. Cor. 4. *Two prisms have the same ratio as the products of their bases and altitudes.*

Ex. 38. Find the volume of a right prism whose altitude is 9 in., and whose base is inclosed by a regular hexagon of side 3 in.

Ex. 39. Express by formulas the total area and the volume of a right prism whose height is h, and whose base is inclosed by a regular hexagon of side e.

Ex. 40. What is the volume and lateral area of a right triangular prism whose altitude is 15 in., and whose base is inclosed by a right triangle whose sides are 6 and 8 in., respectively, and whose hypotenuse is 10 in.?

Ex. 41. What is the volume of the prism whose altitude is 15 in., and whose base is inclosed by a parallelogram with base 8 and altitude 3?

Ex. 42. What is the volume and lateral area of the right prism whose altitude is 18 in., and whose base is inclosed by a regular pentagon whose side is 5 in. and whose apothem is approximately 1.81 inches? (See § 359.)

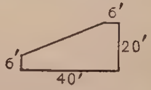
Ex. 43. What is the volume, lateral area, and total area of the right prism whose altitude is 20 in., and whose base is inclosed by a regular hexagon of side 4 in.?

Ex. 44. What is the volume and lateral area of the right prism whose altitude is 24 in., and whose base in inclosed by a quadrilateral $ABCD$ in which $AB = 13$, $BC = 14$, $CD = 12$, $AD = 11$, and $AC = 15$.

Ex. 45. How many tons of soft coal can be stored in a coal bin 15 ft. square and 8 ft. high, if one ton occupies 40 cu. ft.?

Ex. 46. A contractor agreed to dig a cellar at 50¢ per cubic yard. The lot was located upon a hillside so that the depth of the cellar at the back was 8 ft. and in front 5 ft. If the cellar was 38 ft. from the front to back, and was 27 ft. wide, how much did the contractor receive?

Ex. 47. How many cubic yards of concrete are required for a retaining wall 2 ft. thick whose other dimensions are indicated on the adjoining figure?



Ex. 48. What is the volume of the right prism whose altitude is 18 in., and whose base is inclosed by a $\triangle ABC$ in which BC is 9 in., AB is 6 in., and $\angle B = 30°$?

Suggestion. Draw altitude AD to BC. Recall Ex. 157, Book I.

Ex. 49. Repeat Ex. 48 if $\angle B = 60°$.

Ex. 50. Repeat Ex. 48 if $\angle B = 50°$. (Use § 301, and table p. 500.)

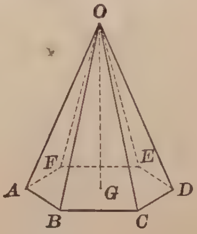
Ex. 51. Find the volume and lateral area of the right prism whose altitude is 24 in., and whose base is inclosed by a rhombus whose diagonals are 6 in. and 8 in.

Ex. 52. Find the volume of an oblique prism whose base is inclosed by a square of side 5 in., if the lateral edge is 10 in. long and if a lateral edge makes an angle of 30° with the perpendicular dropped to the base from its upper extremity.

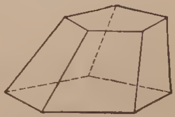
PYRAMIDS

519. A **pyramid** is a polyedron inclosed by three or more triangular faces which have a common vertex, and one other plane face, the **base,** which intersects each of the triangular faces.

The common vertex of the triangular faces is the **vertex** of the pyramid ; the triangular faces are the **lateral faces** ; the edges terminating at the vertex are the **lateral edges** ; the sum of the areas of the lateral faces is the **lateral area** ; the perpendicular from the vertex to the plane of the base is the **altitude.**



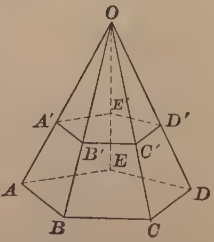
520. A **regular pyramid** is a pyramid whose base is inclosed by a regular polygon, and whose vertex lies in the perpendicular to the base at the center of the base.



521. A **frustum of a pyramid** is the part of a pyramid included between its base and a section parallel to the base.

The **altitude of a frustum** is the perpendicular between the planes of the bases.

522. **Important facts about regular pyramids :**



I. *The lateral edges of a regular pyramid are equal.* (Prove $OA = OB$, etc.)

II. *The lateral faces of a regular pyramid are inclosed by congruent triangles.*

III. *The lateral faces of a frustum of any pyramid are inclosed by trapezoids.*

IV. *The lateral edges of a frustum of a regular pyramid are equal.*

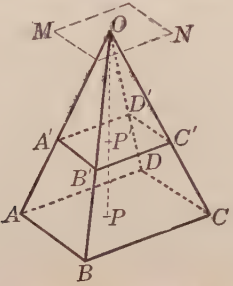
V. *The lateral faces of a frustum of a regular pyramid are inclosed by congruent trapezoids.*

*PROPOSITION VIII. THEOREM

523. *If a pyramid is cut by a plane parallel to the base:*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is similar to the base.*



Hypothesis. Plane $A'C'$, parallel to the base of pyramid $O–ABCD$, intersects faces OAB, OBC, OCD, and ODA in lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, respectively, and altitude OP at P'.

Conclusion. I. $\dfrac{OA'}{OA} = \dfrac{OB'}{OB} = \dfrac{OD'}{OD} = \dfrac{OP'}{OP}$

II. $A'B'C'D' \sim ABCD$.

Plan for proof of I. Through O, pass a plane $MN \parallel ABCD$. Apply § 443.

Proof of II. Statements	Reasons
1. $A'B' \parallel AB$; $B'C' \parallel BC$; $C'D' \parallel CD$; etc.	1. Why?
2. $\angle A'B'C' = \angle ABC$; $\angle B'C'D' = \angle BCD$; etc.	2. Why?
3. $\dfrac{A'B'}{AB} = \dfrac{OB'}{OB} = \dfrac{B'C'}{BC} = \dfrac{OC'}{OC}$; etc.	3. Prove it.
4. $\therefore \dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{C'D'}{CD} = \dfrac{D'A'}{DA}$	4. Why?
5. $\therefore A'B'C'D' \sim ABCD$.	5. § 273.

Note. Two very important corollaries appear on page 368.

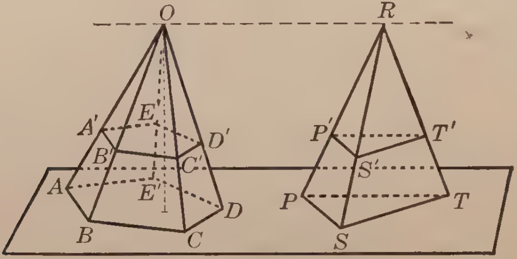
524. Cor. 1. *The area of a section of a pyramid parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

Proof. 1. $\dfrac{\text{Area } A'B'C'D'}{\text{Area } ABCD} = \dfrac{\overline{A'B'}^2}{\overline{AB}^2}$ (§ 336)

2. But $\dfrac{A'B'}{AB} = \dfrac{OA'}{OA} = \dfrac{OP'}{OP}$

3. $\therefore \dfrac{\text{area } A'B'C'D'}{\text{area } ABCD} = \dfrac{\overline{OP'}^2}{\overline{OP}^2}$

525. Cor. 2. *If two pyramids have equal altitudes and equal bases, sections parallel to the bases at equal distances from the vertices are equal.*



Hypothesis. O–$ABCDE$ and R–PST have the common altitude h and *equal* bases. $A'B'C'D'E'$ and $P'S'T'$ are sections of the pyramids parallel to the bases at the distance d from the vertices O and O', respectively.

Conclusion. $A'B'C'D'E' = P'S'T'$.

Proof.

1. $\dfrac{\text{Area } A'B'C'D'E'}{\text{Area } ABCDE} = \dfrac{d^2}{h^2}$; and $\dfrac{\text{area } P'S'T'}{\text{area } PST} = \dfrac{d^2}{h^2}$. (§ 524)

(Complete the proof.)

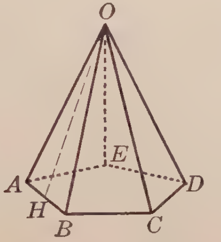
Ex. 53. The altitude of a pyramid is 12 in., and its base is inclosed by a square 9 in. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 8 in.?

526. The slant height of a regular pyramid is the altitude of any lateral face.

Similarly **the slant height of a frustum of a regular pyramid** is the altitude of any lateral face.

*PROPOSITION IX. THEOREM

527. *The lateral area of a regular pyramid equals the perimeter of its base multiplied by one half its slant height.*



Hypothesis. $O\text{–}ABCDE$ is a regular pyramid.

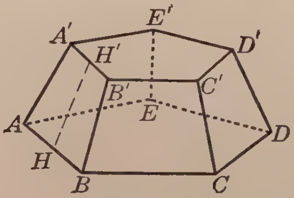
p = the perimeter of its base; s = length of its slant height; and A = the lateral area.

Conclusion. $A = \frac{1}{2} sp.$

Plan. Find the area of each face; add; simplify the result.

PROPOSITION X. THEOREM

528. *The lateral area of a frustum of a regular pyramid equals the sum of the perimeters of the bases multiplied by one half its slant height.*



Hypothesis. AD' is a frustum of a regular pyramid. s = the length of its slant height; p and p' = the perimeters of the lower and upper bases respectively; A = the lateral area.

Conclusion. $A = \frac{1}{2} s(p + p').$

Plan. Recall § 333. Find the area of each face; add.

Ex. 54. What part of the base of a pyramid is the section made by a plane which is parallel to the base and bisects the altitude?

Ex. 55. A plane parallel to the base of a pyramid of altitude 15 in. divides the lateral edge into parts whose ratio is 3 : 2. How long are the two segments of the altitude made by the plane?

Ex. 56. The altitude of a pyramid is 20 in. and its base is inclosed by a regular hexagon whose side is 4 in. What is the area of the section of the pyramid made by a plane 15 in. from the vertex?

Ex. 57. Prove that the perimeter of the mid-section of a frustum of a pyramid equals one half the sum of the perimeters of its bases. (Recall § 161.)

Ex. 58. Express the lateral area of a frustum of a regular pyramid in terms of its slant height and the perimeter of the section mid-way between the two bases.

Ex. 59. What is the lateral area of the regular pentagonal pyramid whose base edge is 6 in., if the slant height is 15 in.?

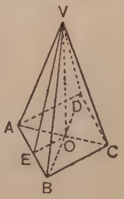
Ex. 60. What is the lateral area of the frustum of the pyramid in Exercise 59, made by a plane which bisects the lateral edges?

Ex. 61. Repeat Exercise 60, if the plane is one third the way from the vertex to the base.

Ex. 62. If VE is the slant height of the regular pyramid V–$ABCD$ and O is the center of the base, prove that $OE \perp AB$.

Ex. 63. (*a*) If, in Ex. 62, $ABCD$ is a square of side 8 in., and $VA = 17$ in., find VE and also VO.

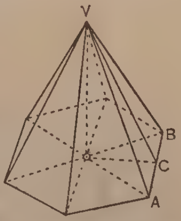
(*b*) Find the lateral area of the pyramid.



Ex. 64. (*a*) What is the slant height of the regular hexagonal pyramid of altitude 12 in., if the base edge is 3 in.?

(*b*) What is the lateral area of the pyramid?

(*c*) What is the lateral area of the frustum made by a plane 3 in. from the base?



Ex. 65. What is the slant height and lateral area of a regular triangular pyramid whose altitude is 18 in., if the base edge is 6 in.?

Ex. 66. The edges of the bases of a frustum of a regular square pyramid are 5 in. and 10 in. respectively, and the altitude is 6 in. Determine the slant height and then the lateral area.

Note. Additional Exercises 31–40, pp. 459–460, can be studied now.

529. A *variable* is a number symbol which has different values during a particular discussion.

A *constant* is a number symbol which has one fixed value during a particular discussion.

A number symbol which has successively the values 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc. is a *decreasing* variable. It ultimately has values which differ from zero by *as little as we please.*

A number symbol which has successively the values 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $\cdots$ is *an increasing variable.* It ultimately has values which differ from 2 by as little as we please.

The length of a burning candle is a decreasing variable, which approaches the length zero. If the candle is allowed to burn out, its length finally becomes zero.

If a child, having some candy, eats one half of it; next eats one half of the first remainder; next one half of the second remainder; and continues, always eating one half of the "remainder"; then the remainder is a decreasing variable. This remainder also approaches zero, *but* theoretically does not become zero, for each time the child eats only one half of the previous remainder.

A limit of a variable is a constant such that the numerical value of the difference between the constant and the variable *becomes and remains less* than any small positive number. We say that a *variable approaches its limit.* If a variable has a limit, it has only one limit. The symbol $\longrightarrow$ means "approaches the limit."

530. Limits Theorems. It can be proved that:

a. If a variable x approaches a finite limit l, then cx, where c is a constant, approaches the limit cl.

b. If two variables are constantly equal and each approaches a finite limit, then their limits are equal.

c. If a variable x approaches a finite limit a, and a variable y approaches a limit b, then:

$(x \pm y)$ approaches the limit $a \pm b$;

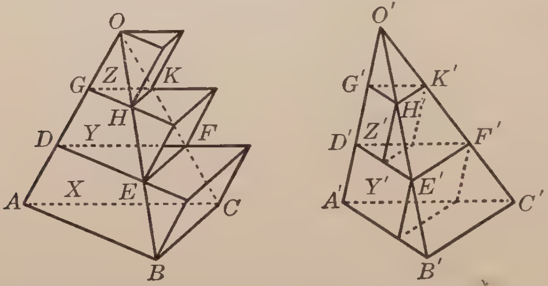
xy approaches the limit $a \cdot b$;

$x \div y$ approaches the limit $a \div b$, provided b is not zero;

and $\sqrt{xy}$ approaches the limit $\sqrt{ab}$.

PROPOSITION XI. THEOREM

531. *Two triangular pyramids having equal altitudes and equal bases are equal.* (See Footnote.)



Hypothesis. $O\text{–}ABC$ and $O'\text{–}A'B'C'$ have equal altitudes and equal bases ABC and $A'B'C'$.

Conclusion. $O\text{–}ABC = O'\text{–}A'B'C'$.

Proof. 1. Place the pyramids with their bases in the same plane, and let h represent their common altitude.

Divide h into 3 equal parts.

Through the points of division pass planes ∥ to the plane of the bases, cutting $O\text{–}ABC$ in sections DEF and GHK, and $O'\text{–}A'B'C'$ in sections $D'E'F'$ and $G'H'K'$.

2. $DEF = D'E'F'$ and $GHK = G'H'K'$ (§ 525)

3. With ABC, DEF, and GHK *as lower bases*, construct prisms X, Y, and Z with their lateral edges equal and ∥ to AD; with $D'E'F'$ and $G'H'K'$ *as upper bases*, construct prisms Y' and Z', with their lateral edges equal and ∥ to $A'D'$.

4. $\therefore Y = Y'$ and $Z = Z'$. (§ 515)

5. Evidently $O\text{–}ABC < X + Y + Z$
and $O'\text{–}A'B'C' > Y' + Z'$.

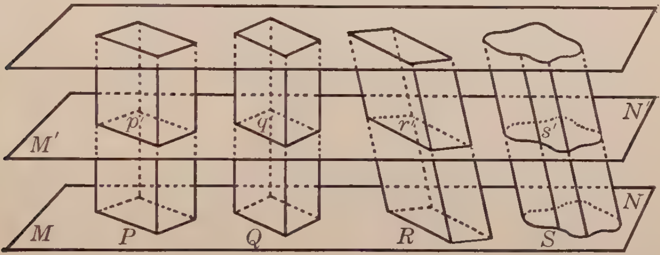
6. $\therefore O\text{–}ABC - O'\text{–}A'B'C' < (X+Y+Z) - (Y'+Z')$. (Ax. 13)
or $O\text{–}ABC - O'\text{–}A'B'C' < X$.

Note. §§ 532 and 533 contain an *interesting optional alternative proof* of § 531.

7. By increasing indefinitely the number of subdivisions of h, the volume of X can be made as small as we please.

8. $\therefore$ O–ABC and O'–$A'B'C'$ must be equal, for if they were unequal, X would be greater than their difference, and this contradicts Step 7.

532. Cavalieri's Theorem. Consider the solids, below, standing upon the plane MN. Assume that every plane $M'N'$, parallel to MN and intersecting the solids, forms in the solids sections of equal area; *i.e.* assume that $p' = q' = r' = s'$.



It is as if each solid consisted of sheets of paper and as if the sheets at the same level above plane MN were of equal size.

If this condition is satisfied, the solids must be between parallel planes.

Two rectangular parallelepipeds standing on the same plane, having equal altitudes and equal bases, are such solids. We know they are equal in volume. (*I.e.* $P = Q$.)

If now $p' = r' = s'$, it is also easy to believe that $P = R = S$.

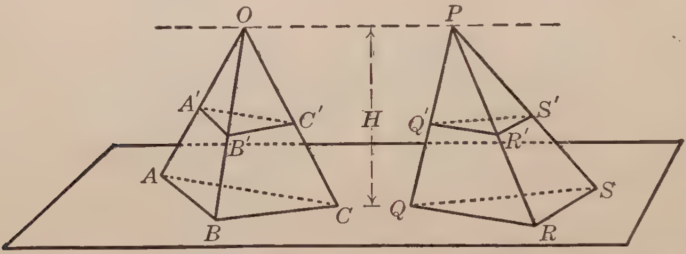
These solids illustrate

Cavalieri's Theorem. *If two solids lie between parallel planes, and if a plane parallel to these parallel planes makes equal sections in these solids no matter where it be drawn between the planes, then the solids have equal volumes.*

PROPOSITION XI′. THEOREM

(Proof based on Cavalieri's Theorem. See Note, p. 372.)

533. *Any two pyramids having equal bases and altitudes are equal.*



Hypothesis. $O\text{–}ABC$ and $P\text{–}QRS$ are two pyramids having the same altitude and equal bases.

Conclusion. $O\text{–}ABC = P\text{–}QRS$.

Proof:

STATEMENTS	REASONS
1. $O\text{–}ABC$ and $P\text{–}QRS$ have the same altitude; hence, they can be placed between parallel planes.	1. Hyp.
2. Let $A'B'C'$ and $Q'R'S'$ be sections of $O\text{–}ABC$ and $P\text{–}QRS$, respectively, made by planes parallel to the bases and at the same distance from the vertices. Then area $A'B'C'$ = area $Q'R'S'$.	2. § 525.
3. ∴ volume $O\text{–}ABC$ = volume $P\text{–}QRS$.	3. § 532.

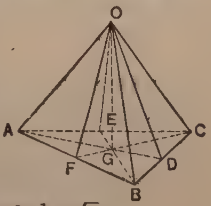
Note. Observe that more is proved in § 533 than in § 532.

Ex. 67. Each side of the base of a regular triangular pyramid is 6, and its altitude is 4. Find its lateral edge, lateral area, and volume. (For volume use § 534.)

Suggestion. Recall § 166, and Note, p. 363.

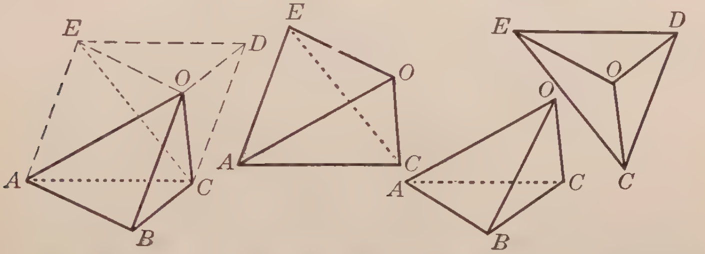
Ex. 68. Find the area of the entire surface and the volume of a triangular pyramid, each of whose edges is 2.

Ex. 69. If all the edges of a regular triangular pyramid are of length l, the altitude equals $\frac{1}{3} l\sqrt{6}$.



*PROPOSITION XII. THEOREM

534. *The volume of a triangular pyramid equals one third the product of its base and altitude.*



Hypothesis. h = the length of the altitude; B = the area of the base; and V = the volume of pyramid O–ABC.

Conclusion. $V = \frac{1}{3}hB$.

Proof. 1. Let EOD–ABC be the triangular prism having base ABC, and its lateral edges equal and parallel to OB.

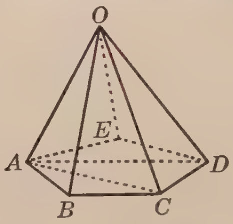
2. Prism EOD–ABC is composed of pyramid O–ABC and pyramid $OACDE$. Separate O–$ACDE$ into pyramids O–ACE and O–ECD, by plane EOC.

3. O–ACE and O–ECD have the same altitude.	3. Why?
4. Base ACE = base ECD.	4. Why?
5. $\therefore$ O–ACE = O–ECD.	5. Why?
6. Pyramids O–ACB and C–EOD have equal altitudes.	6. Why?
7. Base ACB = base EOD.	7. Why?
8. $\therefore$ O–ACB = C–EOD, or O–ECD.	8. Why?
9. $\therefore$ O–ACB = O–ECD = O–ACE.	9. Why?
10. $\therefore$ O–ACB = $\frac{1}{3}$ prism EOD–ABC.	10. Why?
11. The volume of EOD–ABC = $B \times h$.	11. Prove it.
12. $\therefore$ vol. O–ACB = $\frac{1}{3} B \times h$.	12. Why?

Ex. 70. If the base of a pyramid is inclosed by a parallelogram, the plane determined by the vertex of the pyramid and a diagonal of the base divides the pyramid into two equal triangular pyramids.

PROPOSITION XIII. THEOREM

535. *The volume of any pyramid is equal to one third the product of its base and altitude.*



Hypothesis. $O\text{–}ABCDE$ is a pyramid, the length of whose altitude $= h$, the area of whose base $= B$, and whose volume $= V$.

Conclusion. $V = \frac{1}{3} B \cdot h$.

Plan of proof is like that for § 514.

(Proof to be given by the pupil.)

536. Cor. 1. *Two pyramids having equal bases and equal altitudes are equal.* (Omit if § 533 was studied.)

Cor. 2. *Two pyramids having equal altitudes have the same ratio as their bases.*

Cor. 3. *Two pyramids having equal bases have the same ratio as their altitudes.*

Cor. 4. *Two pyramids have the same ratio as the products of their bases and altitudes.*

Ex. 71. Determine the ratio to a given parallelepiped of the pyramid whose lateral edges are the three edges of the parallelepiped which intersect at any one corner.

Ex. 72. Find the volume of a regular square pyramid each side of whose base is 8, and whose altitude is 20.

Ex. 73. Find the volume of a regular hexagonal pyramid each side of whose base is 3, and whose altitude is 5.

Ex. 74. Prove that the lines joining the center of a cube to the four vertices of one face are the edges of a regular square pyramid whose volume is $\frac{1}{6}$ that of the cube.

Ex. 75. How many diagonals does a parallelepiped have?

Ex. 76. How many faces of a right parallelepiped are necessarily inclosed by rectangles?

Ex. 77. Repeat Ex. 76 if the figure is a rectangular parallelepiped.

Ex. 78. From the upper end of a lateral edge of a prism, a perpendicular is dropped to the base, and the lower end is joined to the foot of this perpendicular. If the perpendicular is 18 in. long, and the angle made by the lateral edge and the perpendicular is 30°, how long is the lateral edge?

Ex. 79. What is the lateral area, and the volume of a right prism whose altitude is 15 in. long, and whose base is inclosed by a parallelogram having edges 6 in. and 12 in. long, if the acute angle between these edges is 30°?

Ex. 80. Find the lateral area and the volume of a regular square pyramid, the area of whose base is 100 sq. in., and whose lateral edge is 18 in. long.

Ex. 81. The slant height and lateral edge of a regular square pyramid are 25 and $\sqrt{674}$, respectively. Find its lateral area.

Ex. 82. Express the volume of a pyramid in terms of its altitude and the area of its mid-section parallel to the base.

Ex. 83. Find the volume of a regular square pyramid whose lateral faces are equilateral triangles if the base edge is 12 in. long.

Ex. 84. What is the lateral area, the total area, and the volume of the right prism whose altitude is 15 in., and whose base is inclosed by a regular hexagon whose edge is 4 in.?

Ex. 85. Find the lateral area and the volume of a regular square pyramid whose base edge is 10 in., and whose lateral edge is 16 in.

Ex. 86. What is the lateral area and the volume of a right triangular prism if the altitude is 18 in., and the length of its base edges are 5 in., 8 in., and 10 in.?

Ex. 87. What is the volume of a pyramid whose altitude is 16 in. long and whose base is inclosed by a rhombus whose sides are 6 in. long and whose acute angles are 30°?

Ex. 88. A regular triangular pyramid has a lateral edge of 20 in. and base edge of 24 in. A plane cuts it, parallel to the base and 4 in. from the vertex. What is the lateral area of the frustum formed?

Ex. 89. What is the volume of a regular triangular pyramid whose edges are all 20 in. long?

Note. Additional Exercise 41, page 460, can be studied now.

OPTIONAL TOPICS

Topic A. *The volume of a Prismatoid.*
Topic B. *Frustum of Pyramid* and *Truncated Prism.*
Topic C. *Regular Polyedrons.*

Required as a subsidiary topic by the College Entrance Examination Board.

Topic D. *Similar Polyedrons.*

Required as a subsidiary topic by the C. E. E. B.

OPTIONAL TOPIC A

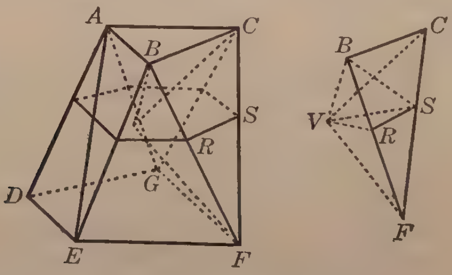
537. A **prismatoid** is a polyedron consisting of two parallel faces called **bases**, and a number of lateral faces inclosed either by triangles, trapezoids, or parallelograms, each of which has a line in one base of the prismatoid and the opposite vertex or base in the other base of the prismatoid. See the figure for § 538.

The **altitude** of the prismatoid is the perpendicular between its bases. The **mid-section** is the section made by the plane parallel to and midway between the bases.

PROPOSITION XIV. THEOREM

538. *If the areas of the lower and upper bases of a prismatoid are B and B′, respectively, the area of the mid-section is M, the length of the altitude is h, and the volume is V, then,*

$$V = \frac{1}{6}h(B + B' + 4M).$$



(Proof on page 379.)

Proof. Through any point V of the mid-section and each edge of the prismatoid, pass planes. These planes divide the prismatoid into pyramid V–ABC, pyramid V–$DEFG$, pyramids like V–BCF, and polyedrons like V–$ABED$.

(*a*) Volume $V\text{–}ABC = \frac{1}{3} \cdot \frac{1}{2} h \cdot B' = \frac{1}{6} hB'$.

(*b*) Similarly, volume $V\text{–}DEFG = \frac{1}{6} hB$.

(*c*) To compute the volume of V–BCF :

1. Draw VR and VS, thus forming $\triangle VRS$, which is a part of the mid-section.
2. In V–BCF and V–RSF :
   Base $BCF = 4 \times$ base RSF. Prove it.
   The altitudes are common.
3. $\therefore V\text{–}BCF = 4 \times (V\text{–}RSF)$. Prove it.
4. But $V\text{–}RSF = \frac{1}{3} \times \frac{1}{2} h \times \triangle VRS$, or $\frac{1}{6} h \times \triangle VRS$.
5. $\therefore V\text{–}BCF = \frac{4}{6} h \times \triangle VRS$.

Similarly, the volume of any triangular pyramid with vertex V whose base is a triangular lateral face is equal to $\frac{4}{6} h$ multiplied by that part of M which is inside the triangular pyramid.

(*d*) V–$ABED$ can be divided into two triangular pyramids by passing a plane through V, A, and E. Hence its volume can be obtained as in part (*c*).

(*e*) Hence, the sum of all pyramids with vertex V, whose bases are lateral faces of the prismatoid, is $\frac{4}{6} hM$.

(*f*) $\therefore$ volume of the prismatoid

$$= \frac{1}{6} hB + \frac{1}{6} hB' + \frac{4}{6} hM$$
$$= \frac{1}{6} h(B + B' + 4M).$$

Ex. 90. Is a prism a special case of a prismatoid? In a prism, what relation is there between B, B', and M? Does the formula of § 538 reduce to the usual formula for the volume of a prism?

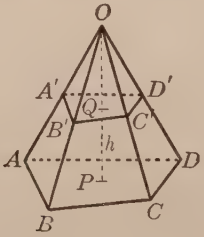
Ex. 91. Answer the same questions for a pyramid.

OPTIONAL TOPIC B

FRUSTUM OF PYRAMID AND TRUNCATED PRISM

PROPOSITION XV. THEOREM

539. *The volume of a frustum of any pyramid is equal to one third of its altitude multiplied by the sum of its upper base, its lower base, and the mean proportional between its bases.*



Hypothesis. V = the volume, B = the area of the lower base, B' = the area of the upper base, and h = the length of the altitude of AC', a frustum of any pyramid $O\text{–}AC$.

Conclusion. $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$.

Proof. 1. Draw altitude OP, cutting $A'C'$ at Q.

2. $$\begin{aligned} V &= \text{vol. } O\text{–}AC - \text{vol. } O\text{–}A'C' \\ &= \tfrac{1}{3}B(h + OQ) - \tfrac{1}{3}B'(OQ) \\ &= \tfrac{1}{3}hB + \tfrac{1}{3}OQ(B - B'). \end{aligned}$$

3. But $B : B' = OP^2 : OQ^2$. § 524

4. $\therefore \sqrt{B} : \sqrt{B'} = OP : OQ$.

5. $\therefore (\sqrt{B} - \sqrt{B'}) : \sqrt{B'} = (OP - OQ) : OQ = h : OQ$. § 258

6. $\therefore OQ(\sqrt{B} - \sqrt{B'}) = h\sqrt{B'}$.

7. Multiplying both members by $\sqrt{B} + \sqrt{B'}$,

$$OQ(B - B') = h(\sqrt{BB'} + B').$$

8. Substituting in step 2 for $OQ(B - B')$ its value from step 7,

$$\begin{aligned} V &= \tfrac{1}{3}hB + \tfrac{1}{3}h(\sqrt{BB'} + B') \\ &= \tfrac{1}{3}h(B + B' + \sqrt{BB'}). \end{aligned}$$

Ex. 92. Find the volume of a frustum of a regular square pyramid, whose base edges are 9 and 5, respectively, and whose altitude is 10.

Ex. 93. Find the volume of a frustum of a regular triangular pyramid, the sides of whose bases are 18 and 6, respectively, and whose altitude is 24.

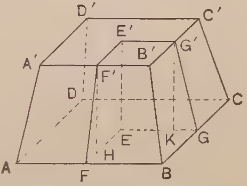
Ex. 94. Find the volume of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 4, respectively, and whose altitude is 12.

Ex. 95. A monument is in the form of a frustum of a regular square pyramid 8 ft. in height, the sides of whose bases are 3 ft. and 2 ft., respectively, surmounted by a regular square pyramid 2 ft. in height, each side of whose base is 2 ft. What is its weight, at 180 lb. to the cubic foot?

Ex. 96. The areas of the bases of a frustum of a pyramid are 12 and 75, respectively, and its altitude is 9. What is the altitude of the pyramid?

Suggestion. Let the altitude of the pyramid $= x$; then $x - 9$ is the $\perp$ from its vertex to the upper base of the frustum.

Ex. 97. The lateral edge of a frustum of a regular square pyramid is 10, and the sides of its bases are 10 and 4, respectively. Find its lateral area and volume.



Ex. 98. Find the lateral edge, lateral area, and volume of a frustum of a regular square pyramid, the sides of whose bases are 17 and 7, respectively, and whose altitude is 12.

Ex. 99. The bases of a frustum of a pyramid are rectangular, having sides of 27 and 15, and 9 and 5, respectively, and the line joining their centers is perpendicular to each base. If the altitude of the frustum is 12, find its lateral area and volume.

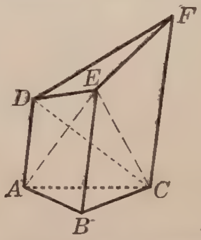
Ex. 100. Find the lateral area and volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 and 6, respectively, and whose lateral edge is 5.

Ex. 101. The altitude and lateral edge of a frustum of a regular triangular pyramid are 8 and 10, respectively, and each side of its upper base is $2\sqrt{3}$. Find its volume and lateral area.

540. A **truncated prism** is formed by the base of a prism, by a plane not parallel to the base which intersects all the lateral faces of a prism, and by the portions of the lateral faces between this plane and the base.

PROPOSITION XVI. THEOREM

541. *A truncated triangular prism is equal to the sum of three pyramids having as common base the lower base of the given prism, and having as their vertices the three vertices of the upper base of the truncated prism.*



Hypothesis. $DEF-ABC$ is a truncated triangular prism.

Conclusion.

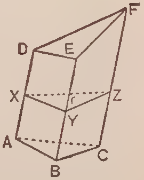
$$DEF-ABC = E-ABC + D-ABC + F-ABC.$$

Proof. 1. Pass planes through A, E, and C, and through D, E, and C, thus dividing $DEF-ABC$ into $E-ABC$, $E-ADC$, and $E-DFC$.

2. $E-ABC$ is one of the required pyramids.
3. In $E-DAC$ and $B-DAC$:
Base DAC is common.
The alt. from E to DAC = the alt. from B to DAC.
4. $\therefore E-DAC = B-DAC.$ Why?
5. But $B-DAC = D-ABC.$
6. $\therefore E-DAC = D-ABC$, the second required pyramid.
7. In $E-DFC$ and $B-AFC$:
Base DFC = base AFC.
The alt. from E to DFC = the alt. from B to AFC.
8. $\therefore E-DFC = B-AFC$
9. But $B-AFC = F-ABC$
10. $\therefore E-DFC = F-ABC$, the third required pyramid.
11. $\therefore DEF-ABC = E-ABC + D-ABC + F-ABC.$

542. Cor. 1. *The volume of a truncated right triangular prism is equal to one third the base multiplied by the sum of the lateral edges.*

543. Cor. 2. *The volume of any truncated triangular prism is equal to one third the area of a right section multiplied by the sum of the lateral edges.*



Suggestions. **1.** Let XYZ be the right section whose area is R.

2. Apply § 542 to $DEF\text{-}XYZ$, and to $ABC\text{-}XYZ$.

3. Prove $DEF\text{-}ABC = \frac{1}{3}R(AD + BE + CF)$.

Ex. 102. Find the volume of a truncated right triangular prism, the sides of whose base are 5, 12, and 13, and whose lateral edges are 3, 7, and 5, respectively.

Ex. 103. Find the volume of a truncated right triangular prism whose lateral edges are 11, 14, and 17, having its base inclosed by an isosceles triangle whose sides are 10, 13, and 13, respectively.

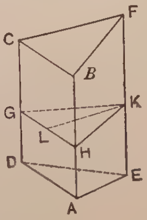
Ex. 104. Find the volume of a polyedron whose base is inclosed by a square having its sides 8 in. long; whose lateral edges are perpendicular to this base; and whose lateral edges are 2, 6, 8, and 4 in. long, respectively.

Suggestion. Pass a plane through two diagonally opposite lateral edges, dividing the solid into two truncated right triangular prisms.

Ex. 105. If $ABCD$ is a rectangle, and EF is any line not in its plane parallel to AB, the volume of the solid bounded by $ABCD$, $ABFE$, $CDEF$, ADE, and BCF, is

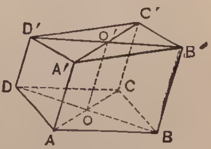
$$\tfrac{1}{6}h \times AD \times (2\,AB + EF),$$

where h is the perpendicular from any point of EF to $ABCD$.



Ex. 106. Prove that the sum of two opposite lateral edges of a truncated parallelepiped is equal to the sum of the other two lateral edges.

Suggestion. Use the Fig. opposite Ex. 107.



Ex. 107. Prove that the volume of a truncated parallelepiped is equal to the area of a right section multiplied by one fourth the sum of the lateral edges.

Suggestion. Use the figure of Ex. 106, omitting AC and $A'C'$. Apply § 543, and Ex. 106.

Note. Additional Exercises 42–53, pp. 460–461, can be studied now.

OPTIONAL TOPIC C

REGULAR POLYEDRONS

544. A **regular polyedron** is a polyedron whose faces are inclosed by congruent regular polygons, and whose polyedral angles are all equal.

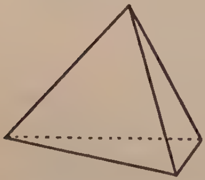
PROPOSITION XVII. THEOREM

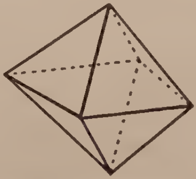
545. *Not more than five regular convex polyedrons are possible.*

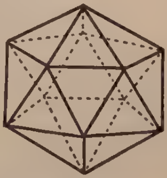
1. *Polyedrons whose faces are inclosed by equilateral* $\triangle$s.

Since each ∠ of an equilateral $\triangle$ is 60°, a convex polyedral ∠ can be formed having 3, 4, or 5, but not more than 5 faces inclosed by equilateral $\triangle$s. (§ 476.) Hence not more than three convex regular polyedrons can be formed having such faces.

These really can be formed. They are pictured below:


REGULAR TETRAEDRON
4 faces


REGULAR OCTAEDRON
8 faces


REGULAR ICOSAEDRON
20 faces

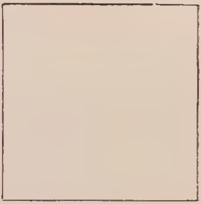
2. *Polyedrons whose faces are inclosed by squares.*

Since each ∠ of a square is 90°, a convex polyedral ∠ can be formed having 3 but not more than 3 faces inclosed by squares. Hence only one regular convex polyedron can be formed having such faces. It is the *cube.*

3. *Polyedrons whose faces are inclosed by regular pentagons.*

Since each ∠ of a regular pentagon is 108°, a convex polyedral ∠ can be formed having 3 but not more than 3 faces inclosed by regular pentagons. Hence only one regular convex polyedron can be formed having such faces.

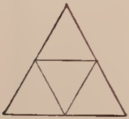
This polyedron is the *dodecaedron*. It has twelve faces.



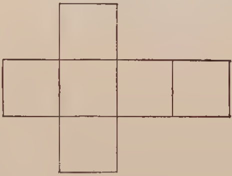
SQUARE

4. *Polyedrons whose faces are inclosed by other regular polygons are impossible,* for if the faces have six or more sides, the sum of the face angles at the vertex of each polyedral ∠ will be > 360°.

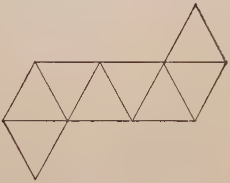
546. Patterns for the five regular polyedrons.



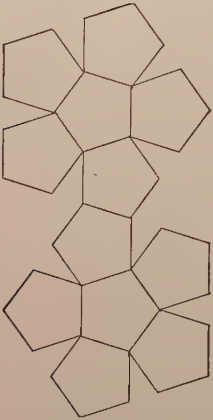
TETRAEDRON



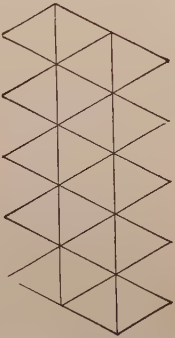
HEXAEDRON



OCTAEDRON



DODECAEDRON



ICOSAEDRON

Ex. 108. What is the sum of the face angles at any vertex of each of the regular polyedrons?

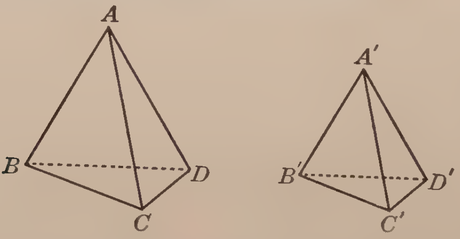
OPTIONAL TOPIC D

SIMILAR POLYEDRONS

547. Two **polyedrons are similar** when they have the same number of faces similar each to each and similarly placed, and have their corresponding polyedral angles equal.

PROPOSITION XVIII. THEOREM

548. *Two tetraedrons are similar when three face triangles including a triedral angle of one are similar, respectively, to three face triangles including a triedral angle of the other, and similarly placed.*



Hypothesis. In tetraedrons $ABCD$ and $A'B'C'D'$
$\triangle ABC \sim \triangle A'B'C'$; $\triangle ACD \sim \triangle A'C'D'$; and
$\triangle ADB \sim \triangle A'D'B'$.

Conclusion. $ABCD \sim A'B'C'D'$.

Proof. 1. From the given similar $\triangle$, we have

$$\frac{BC}{B'C'} = \left(\frac{AC}{A'C'}\right) = \frac{CD}{C'D'} = \left(\frac{AD}{A'D'}\right) = \frac{BD}{B'D'}. \qquad \text{Why?}$$

2. Hence, $\triangle BCD \sim \triangle B'C'D'$. Why?

3. Again, $\angle BAC$, CAD, and DAB are equal, respectively, to $\angle B'A'C'$, $C'A'D'$, and $D'A'B'$. Why?

4. Then, triedral $\angle A\text{–}BCD$ and $A'\text{–}B'C'D'$ are equal.

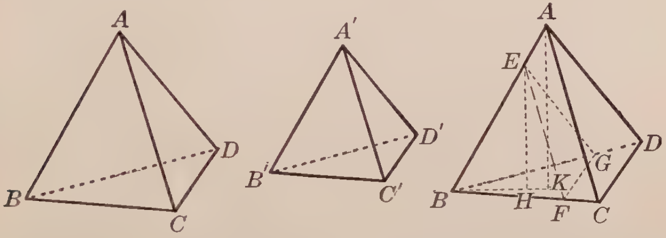
5. Similarly, any two corresponding triedral $\angle$ are equal.

6. Therefore, $ABCD$ and $A'B'C'D'$ are similar.

Ex. 109. If a tetraedron be cut by a plane parallel to one of its faces, the tetraedron cut off is similar to the given tetraedron.

PROPOSITION XIX. THEOREM

549. *Two similar tetraedrons have the same ratio as the cubes of any two corresponding edges.*



Hypothesis. V and V' are the volumes of similar tetraedrons $ABCD$ and $A'B'C'D'$, respectively, A and A', B and B', etc., being corresponding vertices.

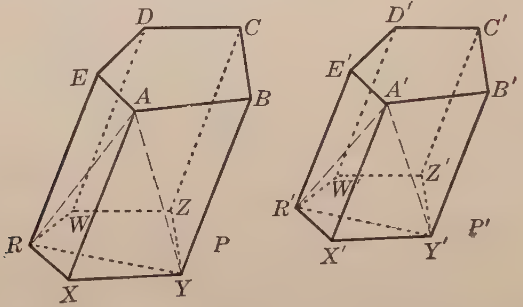
Conclusion. $\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}$.

Proof:

	Statements	Reasons
1.	Place $A'B'C'D'$ on $ABCD$ so that $\angle B'$ coincides with $\angle B$ to which it is equal, taking the position $BEFG$.	1. § 60.
2.	Draw altitudes EH and AK, meeting BCD at H and K, respectively.	2. § 423.
3.	Plane $BAK \perp$ plane BCD.	3. § 459.
4.	$\therefore EH$ lies in plane BAK.	4. § 458.
5.	$\therefore H$ lies on BK.	5. § 413.
6.	$\therefore \frac{EH}{AK} = \frac{BE}{BA} = \frac{B'A'}{BA}$.	6. Prove it.
7.	$\frac{V}{V'} = \frac{\text{area } BCD \times AK}{\text{area } B'C'D' \times EH}$.	7. Cor. 4, § 536.
8.	But $\frac{\text{area } BCD}{\text{area } B'C'D'} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{BA}^2}{\overline{B'A'}^2}$.	8. Prove it, using § 335.
9.	$\therefore \frac{V}{V'} = \frac{\overline{BA}^2}{\overline{B'A'}^2} \times \frac{BA}{B'A'}$, or $\frac{\overline{BA}^3}{\overline{B'A'}^3}$.	9. Ax. 2, § 49.

PROPOSITION XX. THEOREM

550. *Two similar polyedrons can be separated into the same number of tetraedrons, similar each to each and similarly placed.*



Hypothesis. Let P and P' be two similar polyedrons, so lettered that points A, B, C, etc. correspond to A', B', C', etc.

Conclusion. P and P' can be separated into the same number of tetraedrons similar each to each and similarly placed.

Proof. 1. Pass planes through A, R, and Y, and A', R', and Y', forming tetraedrons $A\text{-}RXY$ and $A'\text{-}R'X'Y'$.

2. Triedral $\angle X$ = triedral $\angle X'$. Hyp.

3. $\triangle RXA \sim \triangle R'X'A'$; $\triangle RXY \sim \triangle R'X'Y'$; $\triangle AXY \sim \triangle A'X'Y'$. § 295

4. $\therefore \frac{RA}{R'A'} = \frac{RX}{R'X'} = \frac{AX}{A'X'}$; $\frac{RY}{R'Y'} = \frac{RX}{R'X'} = \frac{AX}{A'X'}$; and $\frac{AY}{A'Y'} = \frac{AX}{A'X'}$. Why?

5. $\therefore \frac{RA}{R'A'} = \frac{RY}{R'Y'} = \frac{AY}{A'Y'}$. Why?

6. $\therefore \triangle RAY \sim \triangle R'A'Y'$. Why?

7. $\angle XRA = \angle X'R'A'$; $\angle XRY = \angle X'R'Y'$; and $\angle YRA = \angle Y'R'A'$. Why?

8. $\therefore$ triedral $\angle R\text{-}AXY$ = triedral $\angle R'\text{-}A'X'Y'$. § 478

Similarly
$\angle A\text{-}RXY = \angle A'\text{-}R'X'Y'$, and $\angle Y\text{-}ARX = \angle Y'\text{-}A'R'X'$.

9. $\therefore$ tetraedron $A\text{-}RXY \sim$ tetraedron $A'\text{-}R'X'Y'$. § 547

10. After tetraedron $A\text{–}RXY$ and $A'\text{–}R'X'Y'$ are removed from P and P', the remaining polyedrons are similar.

For: (*a*) the remaining polyedral ∡ are still equal each to each.
(*b*) the old faces are similar each to each.
(*c*) face $RAY \sim$ face $R'A'Y'$, by proof in Step 6.
(*d*) face $RWZY$ can be proved similar to $R'W'Z'Y'$
(*e*) the new polyedral ∡ are equal each to each.

11. Two similar tetraedrons can be cut off from the remainders described in Step 10, as was done in Step 1, etc.

12. $\therefore P$ and P' can be separated into the same number of tetraedrons, similar each to each and similarly placed.

551. Cor. 1. *The ratio of any two corresponding segments of similar polyedrons equals the ratio of any two corresponding edges (e.g.* $RA : R'A' = AX : A'X'$*).*

552. Cor. 2. *The ratio of the areas of any two similar faces of two similar polyedrons equals the ratio of the squares of any two corresponding edges.*

Thus $\dfrac{\text{area } RAY}{\text{area } R'A'Y'} = \dfrac{\overline{RA}^2}{\overline{R'A'}^2} = \dfrac{\overline{AX}^2}{\overline{A'X'}^2}.$ (By § 336, and § 551.)

553. Cor. 3. *The total areas of two similar polyedrons are to each other as the squares of any two corresponding edges.*

Proof. 1. Let similar polyedrons P and P' have faces F_1, F_2, F_3, etc., and F'_1, F'_2, F'_3, etc., and let e and e' be the lengths of two corresponding edges.

2. Then $\dfrac{F_1}{F'_1} = \dfrac{e^2}{e'^2};\ \dfrac{F_2}{F'_2} = \dfrac{e^2}{e'^2};\ \dfrac{F_3}{F'_3} = \dfrac{e^2}{e'^2};$ etc. § 552

3. $\therefore \dfrac{F_1}{F'_1} = \dfrac{F_2}{F'_2} = \dfrac{F_3}{F'_3},$ etc. Ax. 1, § 49

4. $\therefore \dfrac{F_1 + F_2 + F_3 + \cdots}{F'_1 + F'_2 + F'_3 + \cdots} = \dfrac{F_1}{F'_1}.$ § 296

5. $\therefore \dfrac{\text{total area of } P}{\text{total area of } P'} = \dfrac{e^2}{e'^2}.$ Ax. 1, § 49

Ex. 110. The edges of two similar tetraedrons are 8 in. and 12 in. If the area of the smaller is 240 sq. in., what is the area of the larger?

554. Cor. 4. *The ratio of the volumes of any two similar polyedrons equals the ratio of the cubes of any two corresponding edges.*

Proof. 1. Let e and e' be the two corresponding edges of similar polyedrons P and P'; let P and P' be separated into tetraedrons T_1, T_2, T_3, $\cdots$ and T'_1, T'_2, T'_3, $\cdots$ respectively, such that $T_1 \sim T'_1$, $T_2 \sim T'_2$, etc. (§ 550.)

2. Then $\dfrac{T_1}{T'_1} = \dfrac{e^3}{e'^3}$; $\dfrac{T_2}{T'_2} = \dfrac{e^3}{e'^3}$; etc. By § 549

3. $\therefore \dfrac{T_1}{T'_1} = \dfrac{T_2}{T'_2} = \dfrac{T_3}{T'_3} = \text{etc.}$

4. $\therefore \dfrac{T_1 + T_2 + T_3 \cdots}{T'_1 + T'_2 + T'_3 \cdots} = \dfrac{T_1}{T'_1} = \dfrac{e^3}{e'^3}$. § 296

5. $\therefore \dfrac{P}{P'} = \dfrac{e^3}{e'^3}$. Why?

Ex. 111. The volume of a pyramid whose altitude is 7 in. is 686 cu. in. Find the volume of a similar pyramid whose altitude is 12 in.

Ex. 112. If the volume of a prism whose altitude is 9 ft. is 171 cu. ft., find the altitude of a similar prism whose volume is $50\frac{2}{3}$ cu. ft.

Ex. 113. Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. long, what is the length of the second?

Ex. 114. A pyramid whose altitude is 10 in. weighs 24 lb. At what distance from its vertex must it be cut by a plane parallel to its base so that the frustum cut off may weigh 12 lb.?

Ex. 115. An edge of a polyedron is 56, and the corresponding edge of a similar polyedron is 21. The area of the entire surface of the second polyedron is 135, and its volume is 162. Find the area of the entire surface and the volume of the first polyedron.

Ex. 116. The area of the entire surface of a tetraedron is 147, and its volume is 686. If the area of the entire surface of a similar tetraedron is 48, what is its volume?

Suggestion. Let x and y denote the corresponding edges of the tetraedrons.

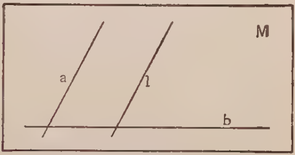
Ex. 117. The area of the entire surface of a tetraedron is 75, and its volume is 500. If the volume of a similar tetraedron is 32, what is the area of its entire surface?

Note. Additional Exercises 54–55, page 462, can be done now.

BOOK VIII

CYLINDERS AND CONES

555. Generating a Surface. If a straight line l moves so that it always intersects a straight line b and is always parallel to a straight line a which intersects b, it can be proved that l always lies in the plane M determined by a and b; also it can be proved that every point in M lies in the line l at some time during its period of movement.

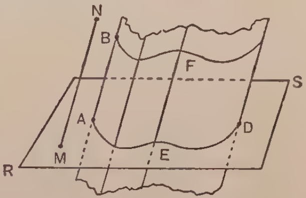


Line l is said to generate the plane M.

By suitable movement of a straight line, the straight line can be made to generate various surfaces.

A cylindrical surface is the surface generated by a moving straight line which always intersects a given plane curve and is parallel to another given straight line not in the plane of the curve.

Thus, if AB moves so as to intersect curve AD of plane RS, and be parallel to MN, not in the plane RS, then AB generates the cylindrical surface BD.



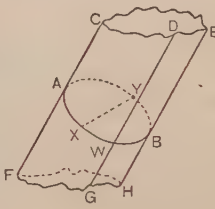
The moving line is called the **generatrix**; the curved line is called the **directrix**. Any position of the generatrix, as EF, is called an **element** of the cylindrical surface.

556. *All the elements of a cylindrical surface are parallel,* by § 555.

557. *Through a point of a cylindrical surface there is only one element.*

For, if there were two elements through F, each would be parallel to AB, and that is impossible.

558. Convex closed cylindrical surfaces alone will be studied in this text. Such surfaces are generated *when the directrix is a closed plane curve such that a straight line or a plane cuts it in two and in only two points;* as directrix *AXBY*.

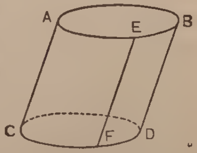


A plane cannot contain more than two elements of a convex closed cylindrical surface, for if it did it would cut the directrix in more than two points.

A convex closed cylindrical surface separates from surrounding space a portion of space, infinite in extent.

559. A **right section of a cylindrical surface** is the intersection of the surface and a plane perpendicular to one and therefore to all the elements of the surface.

560. A **convex cylinder** or **cylinder** is the solid bounded by part of a convex closed cylindrical surface and the parts of two parallel planes, each of which intersects all the elements of the surface.

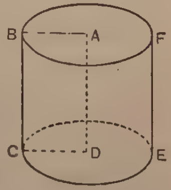


The parts of the parallel planes are the **bases** and the part of the cylindrical surface between the parallel planes is the **lateral surface** of the cylinder.

The **total surface** of the cylinder consists of its bases and its lateral surface.

The **altitude** of a cylinder is the perpendicular between the planes of its two bases.

561. Kinds of cylinders. A **right cylinder** is a cylinder whose bases are perpendicular to one, and hence to all its elements.



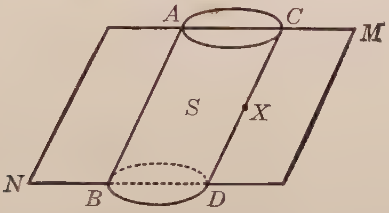
A **circular cylinder** is a cylinder whose base is enclosed by a circle.

Cylinder *BCEF* is a right circular cylinder.

Ex. 1. All the elements of a cylinder are equal. In Figure, § 560, prove $AC = BD$, after drawing AB and CD.

PROPOSITION I. THEOREM

562. *If a plane through an element of a convex cylinder intersects its surface in a point which is not on that element, the intersection of the plane and the total surface of the cylinder is a parallelogram.*



Hypothesis. Plane MN intersects convex cylinder S in element AB and point X on the surface but not on AB. A and B are in the upper and lower bases respectively.

Conclusion. The intersection of MN and the total surface of S is a ▱.

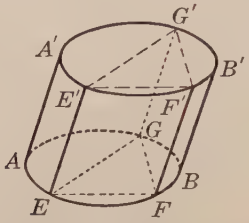
Proof:

Statements	Reasons
1. Let CD be the element of S through X, cutting the upper and lower bases at C and D respectively.	**1.** § 560.
2. $\therefore AB \parallel CD$.	**2.** § 556.
3. AB and CD lie in only one plane, $ABCD$.	**3.** § 410; 4.
4. Plane M coincides with plane $ABCD$, since AB and X lie in plane $ABCD$.	**4.** § 410; 2.
5. $\therefore CD$ lies in plane M.	**5.** Step 4.
6. M cannot cut the lateral surface of S in any other points, for if it did, it would contain the elements through these points, and that is impossible.	**6.** By steps 1–4; and § 558.
7. AC and BD lie in M and on surface of S.	**7.** § 408, 1.
8. $ABDC$ is ▱.	**8.** Prove it.

Ex. 2. If a rectangle revolves about one of its sides, as an axis, it generates a right circular cylinder, or a *cylinder of revolution.* (See Figure § 561.)

PROPOSITION II. THEOREM

563. *The bases of a cylinder are congruent.*



Hypothesis. AB' is any cylinder, with bases $A'B'$ and AB.

Conclusion. Base $A'B' \cong$ base AB.

Proof:

	STATEMENTS	REASONS
1.	Let E' and F' be two particular points of curve $A'B'$ and G' any other point. Draw the elements EE', FF', and GG', cutting the base AB at E, F, and G. Draw $E'G'$, $G'F'$, $E'F'$, EG, GF, and EF.	1. § 560.
2.	EE', FF', and GG' are parallel.	2. § 556.
3.	$E'F' \parallel EF$; $E'G' \parallel EG$; $G'F' \parallel GF$.	3. Prove it.
4.	$\triangle E'G'F' \cong \triangle EGF$.	4. Prove it.
5.	$\therefore$ base $A'B'$ can be superposed on base AB so that E', F', and G', will fall on E, F, and G, respectively.	5. § 59.
6.	$\therefore$ every pt. G' of $A'B'$ falls on a pt. of AB. Also every pt. of AB falls on a pt. of $A'B'$.	6. G' was any point of $A'B'$.
7.	$\therefore$ base $ABC \cong$ base $A'B'C'$.	7. § 59.

564. Cor. *A section of a circular cylinder made by a plane parallel to the base is inclosed by a circle.*

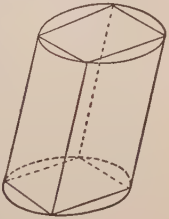
Ex. 3. Prove that a line parallel to the elements of a circular cylinder from the center of one base intersects the other base at its center.

565. The **axis of a circular cylinder** joins the centers of its two bases and is parallel to its elements.

MEASURING THE CYLINDER

566. In **measuring the cylinder**, difficulties are encountered which are like those met in measuring a circle, for the ratio of a cylindrical surface to an ordinary unit of surface measure cannot have meaning in the usual sense since they are not magnitudes of the same kind.

567. A **prism is inscribed in a cylinder** when its lateral edges are elements of the cylinder and its bases are in the planes of the bases of the cylinder. The polygons inclosing the bases of the prism are inscribed in the boundaries of the bases of the cylinder.



568. Application of limits to a cylinder. Inscribe in a circular cylinder a prism having its base inclosed by a regular polygon, such as a square; then inscribe a second prism whose base is inclosed by a regular polygon having double the number of sides; imagine that this process is continued indefinitely. Pass a plane forming right sections of the cylinder and the prisms.

It is obvious that the prisms come nearer and nearer to occupying the same space as the cylinder. Hence:

(*a*) The **volume of a cylinder** is defined to be the limit of the volume of an inscribed prism with base inclosed by a regular polygon as the number of faces of the prism is increased indefinitely.

(*b*) The **lateral area of a cylinder** is defined to be the limit of the lateral area of an inscribed prism with base inclosed by a regular polygon as the number of faces of the prism is increased indefinitely.

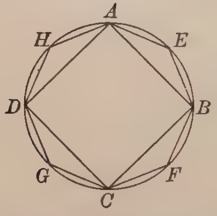
(*c*) The **length of a right section of the lateral surface** of a cylinder is defined to be the limit of the length of the right section of the lateral surface of the inscribed prism.

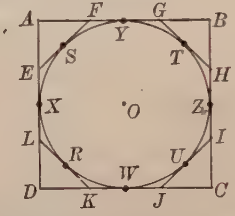
The gist of this section is that a prism having an infinite number of faces is approximately a cylinder.

569. The College Entrance Examination Board recommends that proofs by the theory of limits be, *at least, read* and *understood.* As a basis for them, the following facts about circles are summarized:

570. Application of limits to circles and plane curves.

(*a*) The circumference of a circle is defined to be the limit of the perimeter of a regular inscribed polygon as the number of sides is indefinitely increased.





(*b*) It is proved that the perimeter of *any regular* circumscribed polygon (as well as inscribed polygon) approaches the circumference of the circle as limit if the number of sides is indefinitely increased.

(*c*) The length of *any* simple closed curve lying in a plane is defined to be the limit of the perimeter of any inscribed polygon when the number of sides of the polygon is increased indefinitely in such manner that every side approaches the limit zero.

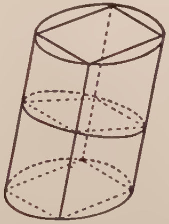
(*d*) The area of a circle is defined to be the limit of the area of any *regular* inscribed polygon as the number of sides is increased indefinitely.

(*e*) It is proved that the area of any *regular* circumscribed polygon approaches the area of the circle as limit as the number of sides is increased indefinitely.

(*f*) It can be proved that the area of *any* inscribed or circumscribed polygon approaches the area of the circle as limit if the number of sides is increased indefinitely in such manner that the length of each side approaches the limit zero.

PROPOSITION III. THEOREM

571. *The lateral area of a circular cylinder equals the perimeter of a right section multiplied by the length of an element.*



Hypothesis. A = the lateral area, p = the perimeter of a right section, e = the length of an element of a cylinder.

Conclusion. $A = e \times p$.

Proof:	Statements	Reasons
1.	Inscribe in the cylinder a prism whose base is inclosed by a regular polygon. Let A' = its lateral area, and p' = the perimeter of a right section of the prism.	1. § 567.
2.	$A' = e \times p'$.	2. § 495.
3.	If the number of faces of the prism is increased indefinitely, $A' \longrightarrow A$ and $p' \longrightarrow p$.	3. (*a*) § 568, *b*. (*b*) § 568, *c*.
4.	$\therefore e \times p' \longrightarrow e \times p$.	4. § 530, *a*.
5.	$\therefore A = e \times p$.	5. § 530, *b*.

572. Cor. 1. *The lateral area of a right circular cylinder equals the circumference of its base multiplied by the length of the altitude.*

573. Cor. 2. *If r = the length of the radius of the base, h = the length of the altitude, A = the lateral area,* and *T = the total area of a right circular cylinder,* then

(*a*) $A = 2\,\pi rh$. (*b*) $T = 2\,\pi r^2 + 2\,\pi rh$, or $2\,\pi r(r + h)$.

Note. If the cylinder is oblique, a right section is an ellipse.

PROPOSITION IV. THEOREM

574. *The volume of a circular cylinder equals the area of its base multiplied by the length of its altitude.*

Hypothesis. V = the volume, B = the area of the base, and h = the length of the altitude of a circular cylinder.

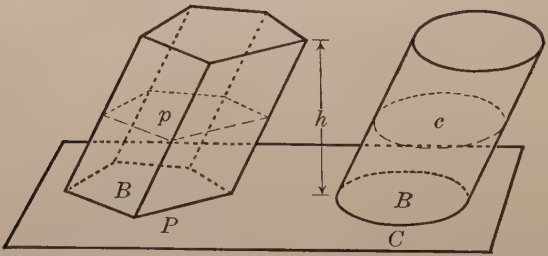
Conclusion. $V = h \times B.$

Proof:

	STATEMENTS	REASONS
1.	Inscribe in the cylinder a prism having its base inclosed by a regular polygon. Let V' = its volume and B' = the area of its base. Then $V' = h \times B'$.	1. § 514.
2.	If the number of faces of the prism be increased indefinitely, $V' \longrightarrow V$ and $B' \longrightarrow B$.	2. (*a*) § 568, *a*. (*b*) § 570, *d*.
3.	$\therefore h \times B' \longrightarrow h \times B.$	3. § 530, *a*.
4.	$\therefore V = h \times B.$	4. § 530, *b*.

575. Cor. *If* r = *the radius of the base of a circular cylinder*, then $V = \pi r^2 h$.

576. Optional proof of § 574, by Cavalieri's Theorem.



1. Let the base and altitude of cylinder C and of prism P be B and h respectively.

Let p and c be sections of P and C respectively at the same distance from the bases. Then $p = c$.

2. $\therefore$ volume of C = volume of P. § 532

3. $\therefore$ volume of $C = h \times B$.

Ex. 4. Find the lateral area of a right circular cylinder whose altitude is 16 and the diameter of whose base is 8.

Ex. 5. Find the total area of a cylinder of revolution whose altitude is 15 and the radius of whose base is 3.

Ex. 6. Determine the cost at 35¢ per square yard of painting the vertical surface and top of a gas holder whose diameter is 30 ft. and whose height is 20 ft.

Ex. 7. How many square feet of tin are required to make 15 sections of hot-air furnace pipe 10 in. in diameter and 30 in. in length?

Ex. 8. What is the cost of digging a dry well 8 ft. in diameter and 15 ft. deep at 50¢ per cubic yard?

Ex. 9. What is the capacity, in gallons, of a water tank 12 ft. in length and 42 in. in diameter, estimating $7\frac{1}{2}$ gal. of water to a cubic foot?

Ex. 10. How many cubic feet of metal are there in a hollow cylindrical tube 18 ft. long, whose outer diameter is 8 in. and whose thickness is 1 in.?

Ex. 11. Determine the number of cubic yards of concrete required for the wall and floor of a circular cistern 6 ft. in outside diameter, and 12 ft. deep, if the walls and floor are 8 in. thick.

Ex. 12. Determine the diameter of a 6 bbl. water reservoir having the form of a right circular cylinder if the altitude is 8 ft.
(2 bbl. = 63 gal.; 1 cu. ft. contains $7\frac{1}{2}$ gal.).

Ex. 13. What is the total area of a right circular cylinder whose altitude is 15 in. and whose base diameter is 5 in.?

Ex. 14. How many gallons will be contained in a right circular cylindrical water tank if the height is 5 ft. and the diameter is 24 in.?

Ex. 15. How many gallons of fuel oil can be stored in a right circular cylindrical tank 72 in. long and 52 in. in diameter?

Ex. 16. What is the number of square feet of radiating surface on a steam pipe 2 in. in outside diameter, and 20 ft. long?

Ex. 17. Prove that the locus of points in space at a given distance from a given line is the circular cylindrical surface whose axis is the given line and the radius of whose right section is the given distance.

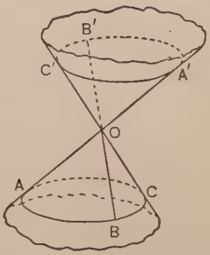
Ex. 18. (*a*) How long is the altitude of a circular oblique cylinder whose elements are 15 in. long, if any element makes an angle of 60° with the perpendicular dropped from its upper end to the base of the cylinder?

(*b*) What is the volume of this cylinder if the radius of its base is 3 in.?

Note. Additional Exercises 53–67, pp. 462–3, can be studied now.

CONES

577. A **conical surface** is the surface generated by a moving straight line, which always intersects a given plane curve and always passes through a given point not in the plane of the curve.



Thus, line OA moves so as always to intersect plane curve ABC, and always passes through point O, not in the plane of the curve.

The moving line is the **generatrix**, and the curve, the **directrix.**

The given point is the **vertex**, and any position of the generatrix, as BOB', is called an **element** of the surface.

If the generatrix is infinite in length, it generates two surfaces of infinite extent, $O\text{–}A'B'C'$ and $O\text{–}ABC$, called the **upper and lower nappes** of the conical surface.

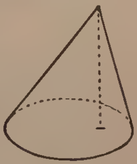
578. *Through every point of a conical surface, excepting the vertex, there is only one element.* (§ 5, *a.*)

579. **A convex closed conical surface** is generated *when the directrix is a closed plane curve such that a straight line or plane will intersect it in two and in only two points.*

A plane cannot contain more than two elements of a convex closed conical surface.

580. *One nappe of a convex closed conical surface separates from surrounding space a portion of space, infinite in extent.*

581. A **cone** is a solid bounded by a portion of one nappe of a convex closed conical surface and that part of a plane which cuts all the elements of the surface and lies within the surface.



The part of the plane is the **base** of the cone, and the part of the conical surface is the **lateral surface** of the cone.

The **altitude** of a cone is the perpendicular from its vertex to the plane of the base.

582. A **circular cone** is a cone whose base is inclosed by a circle.

The **axis of a circular cone** is the straight line drawn from the vertex to the center of the base.

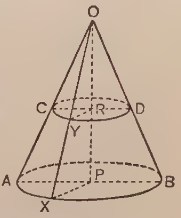
A **right circular cone** is a circular cone whose axis is perpendicular to its base.

A **frustum of a cone** is the part of a cone included between the base and a plane parallel to the base. The base of the cone is called the *lower base,* and the section of the cutting plane the *upper base,* of the frustum.

The **altitude of a frustum of a cone** is the perpendicular between the planes of the bases.

583. *The elements of a right circular cone are equal.*

584. *The elements of a frustum of a right circular cone are equal.* (Prove $AC = XY$.)



585. As a consequence of § 583, the **slant height of a right circular cone** is the length of an element; as OX.

The **slant height of a frustum of a right circular cone** is the length of the part of the slant height of the right circular cone between the bases of the frustum; as XY.

Ex. 19. If a right triangle be revolved about one of its legs as an axis, it generates a right circular cone, called *a cone of revolution.*

Ex. 20. Find, correct to tenths, the slant height of a right circular cone whose altitude is 12 in. and the radius of whose base is 4 in.

Ex. 21. Find the slant height of a right circular cone whose altitude is h and the radius of whose base is r.

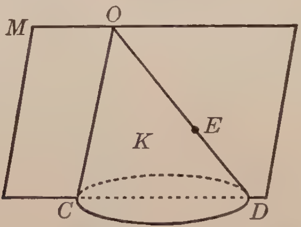
Ex. 22. The radii of the upper and lower bases of the frustum of a right circular cone are 2 in. and 6 in. respectively; the altitude of the frustum is 10 in. Determine the slant height correct to tenths.

Ex. 23. Find the altitude of a right circular cone whose slant height is 15 and the radius of whose base is 5 in.

Ex. 24. In Ex. 23, what would be the slant height of the frustum made by a plane parallel to the base and 4 in. from the vertex?

PROPOSITION V. THEOREM

586. *If a plane passes through an element of a (convex) cone and through at least one other point of the surface of the cone, the intersection with the total surface of the cone is a triangle.*



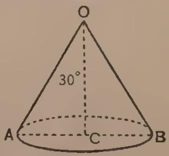
Hypothesis. Plane M, passing through element OC of cone K intersects the surface again in point E, not in OC.

Conclusion. The intersection of M and the total surface of K is a triangle.

Proof: STATEMENTS	REASONS
1. Element OE, through E, lies in M.	1. Why?
2. M cannot contain any other points of the lateral surface of K, for, if it did, it would contain also the elements through those points, and that is impossible.	2. By Step 1, and § 579.
3. OE and OC meet the base in D and C.	3. § 581.
4. CD lies in M and in the base.	4. Why?
5. $\therefore \triangle OCD$ is the intersection of M and K.	5. § 412.

Ex. 25. Prove that the intersection of the surface of a right circular cone and a plane through an element is an isosceles triangle.

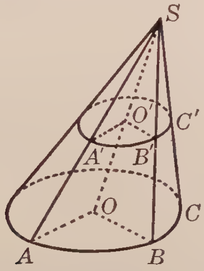
Ex. 26. If the elements of a right circular cone make 30° angles with the altitude, and the radius of the base is 3 in., how long is the slant height and the altitude?



Ex. 27. If the slant height of a right circular cone is 12 in., and if an element makes an angle of 45° with the altitude, what is the radius of the base?

*PROPOSITION VI. THEOREM

587. *The section of the lateral surface of a circular cone made by a plane parallel to the base is a circle.*



Hypothesis. $A'B'C'$ is the section of circular cone $S\text{–}ABC$, made by a plane $\parallel$ base ABC whose center is O.

Conclusion. $A'B'C'$ is a circle.

Proof:

STATEMENTS	REASONS
1. Draw axis OS intersecting plane $A'B'C'$ at O'.	1. § 582.
2. Let A' be one pt., and B' any other pt. of $A'B'C'$. Draw $SA'A$, $SB'B$, OA and OB.	2. § 578; § 581.
3. $\therefore A'O' \parallel AO$ and $B'O' \parallel BO$.	3. Why?
4. $\therefore \frac{A'O'}{AO} = \frac{O'S}{OS}$; and $\frac{B'O'}{BO} = \frac{O'S}{OS}$.	4. Prove it.
5. $\therefore B'O' = A'O'$.	5. § 254.
6. $\therefore$ all points of $A'B'C'$ are at the distance $A'O'$ from O' and $A'B'C'$ is a $\odot$.	6. B' was any pt. on $A'B'C'$.

588. Cor. *The area of a section of a circular cone, made by a plane parallel to the base, is to the base as the square of its distance from the vertex is to the square of the altitude.*

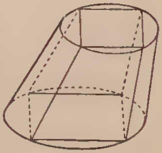
For, if d is the distance and h is the altitude,

$$\frac{\text{area of } \odot\, O'}{\text{area of } \odot\, O} = \frac{\overline{O'A'}^2}{\overline{OA}^2} \text{ (§ 381); and } \frac{\overline{O'A'}^2}{\overline{OA}^2} = \frac{\overline{O'S}^2}{\overline{OS}^2} = \frac{d^2}{h^2}.$$

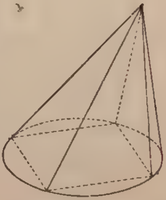
$$\frac{\text{area of } \odot\, O'}{\text{area of } \odot\, O} = \frac{d^2}{h^2}, \text{ or } \left(\frac{d}{h}\right)^2.$$

MEASURING THE CONE

589. A pyramid is inscribed in a cone when its lateral edges are elements of the cone and the bounding polygon of the base of the pyramid is inscribed in the boundary of the base of the cone.



590. A frustum of a pyramid is inscribed in a frustum of a cone when its lateral edges are elements of the frustum of the cone, and the boundaries of its bases are inscribed in the boundaries of the bases of the frustum of the cone.



591. Application of limits to the cone. Inscribe in a circular cone a pyramid *having a base inclosed by a regular polygon*; then inscribe a second pyramid whose base is inclosed by a regular polygon having double the number of sides; imagine that this process is continued indefinitely. Then :

(*a*) The **volume of a cone** is defined to be the limit of the volume of the inscribed pyramid as the number of faces is increased indefinitely.

(*b*) The **lateral area of a cone** is defined to be the limit of the lateral area of the inscribed pyramid as the number of faces is increased indefinitely.

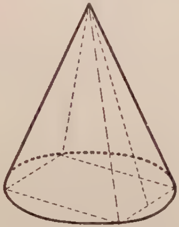
(*c*) It can be proved that the **slant height of a right circular cone** is the limit of the slant height of the inscribed pyramid as the number of faces is increased indefinitely.

(*d*) The **volume and lateral area of a frustum of a cone** are defined to be the limits of the volume and lateral area respectively of the frustum of an inscribed regular pyramid, as the number of faces is increased indefinitely ; also, the slant height of a frustum of a right circular cone can be proved to be the limit of the slant height of the frustum of an inscribed regular pyramid as the number of faces is increased indefinitely.

7, 8.

PROPOSITION VII. THEOREM

592. *The lateral area of a right circular cone equals the circumference of its base multiplied by one half its slant height.*



Hypothesis. A = the lateral area, c = the circumference of the base, and s = the slant height of a right circular cone.

Conclusion. $A = \frac{1}{2} cs$.

Proof:

STATEMENTS	REASONS
1. Inscribe in the cone a pyramid whose base is inclosed by a square. Let A' = the lateral area, p' = the perimeter of the base, and s' = the slant height of the inscribed pyramid.	1. § 589.
2. $A' = \frac{1}{2} p' \times s'$	2. § 527.
3. Increase indefinitely the number of faces of the pyramid, by successively doubling the number of sides of its base. Then $A' \longrightarrow A$; $p' \longrightarrow c$; and $s' \longrightarrow s$.	3. § 591, *b*. § 570, *a*. § 591, *c*.
4. $\therefore \frac{1}{2} p' \times s' \longrightarrow \frac{1}{2} cs$.	4. § 530, *c*.
5. $\therefore A = \frac{1}{2} cs$.	5. § 530, *b*.

593. Cor. *If A denotes the lateral area, T the total area, s the slant height, and r the radius of the base of a right circular cone,* then

$$A = 2\pi r \times \tfrac{1}{2} s, \text{ or } \pi rs.$$

Also, $$T = \pi rs + \pi r^2, \text{ or } \pi r(s + r).$$

Ex. 28. Find the lateral area and the total area of the right circular cone, the radius of whose base is 7 in. and whose slant height is 25 in.

PROPOSITION VIII. THEOREM

594. *The volume of a circular cone equals the area of its base multiplied by one third the length of its altitude.*

Hypothesis. V = the volume, B = the area of the base, and h = the length of the altitude of a circular cone.

Conclusion. $V = \frac{1}{3} Bh$

Proof:	STATEMENTS	REASONS
1.	Inscribe in the cone a pyramid whose base is inclosed by a square. Let V' = its volume, B' = the area of its base, and h = the length of its altitude.	**1.** § 589.
2.	$V' = \frac{1}{3} B' \times h$	**2.** Why?
3.	Increase indefinitely the number of faces of the pyramid by successively doubling the number of sides of its base. Then $V' \longrightarrow V$; $B' \longrightarrow B$.	**3.** § 591, *a.* § 570, *d.*
4.	$\therefore \frac{1}{3} B' \times h \longrightarrow \frac{1}{3} B \times h$	**4.** § 530, *a.*
5.	$\therefore V = \frac{1}{3} Bh.$	**5.** § 530, *b.*

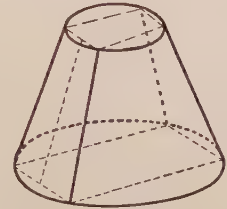
595. Cor. *If V denotes the volume, h the altitude, and r the radius of the base of a circular cone, $V = \frac{1}{3} \pi r^2 h$.*

596. Optional proof of § 594, by Cavalieri's Theorem :

1.	Let pyramid P and cone C have equal bases of area B, and altitude h. Let p and c be sections of P and C, respectively, at the distance d from their vertices.	
2.	$\frac{\text{area of } p}{B} = \frac{d^2}{h^2}$ and $\frac{\text{area of } c}{B} = \frac{d^2}{h^2}$.	§ 524 § 588
3.	$\therefore$ area of p = area of c.	§ 254
4.	$\therefore$ volume of C = volume of P.	§ 532
5.	$\therefore$ volume of $C = \frac{1}{3} Bh.$	Ax. 2

PROPOSITION IX. THEOREM

597. *The lateral area of a frustum of a right circular cone equals the sum of the circumferences of its bases, multiplied by one half its slant height.*



Hypothesis. A = the lateral area, c and c' = the circumferences of the lower and upper bases respectively, and s = the slant height of a frustum of a right circular cone.

Conclusion. $A = \frac{1}{2}s(c + c')$.

Proof:

STATEMENTS	REASONS
1. If A' = the lateral area, p and p' = the perimeters of the lower and upper bases, and s' = the slant height of the frustum of a regular pyramid inscribed in the frustum of the cone, then $A' = \frac{1}{2}s'(p + p')$.	1. § 590. § 528.
2. Increase indefinitely the number of faces of the frustum of the regular inscribed pyramid. Then $A' \longrightarrow A$; $p' \longrightarrow c'$; $p \longrightarrow c$; $s' \longrightarrow s$.	2. § 591, *d*.
3. $\therefore \frac{1}{2}s'(p + p') \longrightarrow \frac{1}{2}s(c + c')$.	3. § 530, *c*.
4. $\therefore A = \frac{1}{2}s(c + c')$.	4. § 530, *b*.

598. Cor. 1. *If A denotes the lateral area, s the slant height, and r and r' the radii of the bases of a frustum of a right circular cone,* $A = (2\pi r + 2\pi r') \times \frac{1}{2}s$, or $\pi s(r + r')$.

599. Cor. 2. *The lateral area of a frustum of a right circular cone is equal to the circumference of a section midway between the bases, multiplied by the slant height.*

Ex. 29. Determine the volume of the right circular cone, the radius of whose base is 6 in. and whose altitude is 25 in.

Ex. 30. Determine the volume of the solid generated when a right triangle of base 4 and altitude 15 revolves around its altitude.

Ex. 31. Determine the ratio of the volumes of a circular cone and a circular cylinder having the same base and altitude.

Ex. 32. How many square yards of canvas are required for a circus tent having the form of a right circular cylinder, surmounted by a right circular cone, if the diameter of the tent is 100 ft., the height of the vertical wall 15 ft., and the height of the highest point of the tent 50 ft.?

Ex. 33. The diameter of the base of a right circular cone is 10 in. and its slant height is 20 in. Determine its lateral and total area, and its volume.

Ex. 34. Prove that the axis of a circular cone passes through the center of every section parallel to the base.

Ex. 35. Prove that the radii of the upper and lower bases of a frustum of any cone have the same ratio as the distances of the bases from the vertex of the cone.

Ex. 36. What is the volume of the right circular cone whose slant height is 20 ft. and whose altitude is 16 ft.?

Ex. 37. What is the lateral area and the volume of the right circular cone whose altitude is 15 in. and the circumference of whose base is 10π in.?

Ex. 38. (*a*) What is the volume of the cone whose diameter is 8 in. and whose altitude is 16 in.?

(*b*) What is the volume of the cone cut off from this given cone by a plane which is parallel to the base and 8 in. from the vertex?

(*c*) What is the volume of the frustum of the original cone lying below this plane?

Ex. 39. Let $ABCD$ be a rectangle, and XY be $\perp$ DC and AB, through point O, the intersection of the diagonals. Assume that rectangle $ABCD$ is revolved around XOY as axis.

(*a*) What is the volume of the cylinder generated by $ABCD$?

(*b*) What is the volume of the cones generated by $\triangle AOB$ and $\triangle DOC$?

Ex. 40. Let $O\text{–}ABCD$ be a right circular cone with altitude 18 in. and radius of base 5 in. Inscribe in it a square pyramid.

(*a*) Compare the volume of the cone with the volume of the pyramid.

(*b*) Compare the lateral area of the cone with that of the pyramid.

Note. Additional Exercises 68–75, pp. 463–4, can be studied now.

OPTIONAL TOPICS

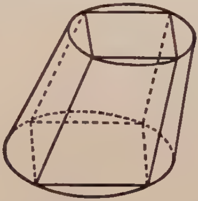
Topic A. The volume of a frustum of a cone.
Topic B. Similar cylinders and cones of revolution.
Topic C. Planes tangent to cylinders and cones.

Topics B and C are included as subsidiary in the list of the College Entrance Examination Board.

OPTIONAL TOPIC A

PROPOSITION X. THEOREM

600. *The volume of a frustum of a circular cone equals the sum of its bases and the mean proportional between them, multiplied by one third its altitude.*



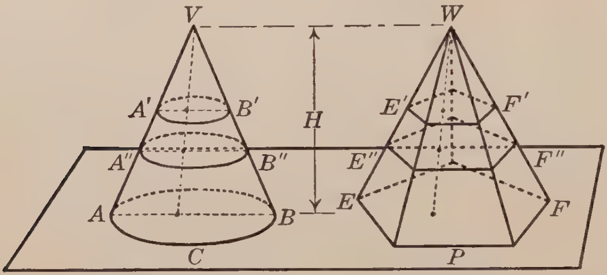
Hypothesis. V = the volume, B and B' = the areas of the lower and upper bases, respectively, and h = the length of the altitude of a frustum of a circular cone.

Conclusion. $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$.

Proof:

STATEMENTS	REASONS
1. Inscribe in the frustum of the cone a frustum of a pyramid having its base inclosed by a square. Let v = its volume, and b and b' = the areas of its lower and upper bases respectively. Then $v = \frac{1}{3}h(b + b' + \sqrt{bb'})$.	1. § 539.
2. Successively double the number of faces of the frustum of the pyramid. Then $v \longrightarrow V$; $b \longrightarrow B$; and $b' \longrightarrow B'$.	2. § 591, *d*.
3. $\frac{1}{3}h(b + b' + \sqrt{bb'}) \longrightarrow \frac{1}{3}h(B + B' + \sqrt{BB'})$.	3. § 530, *c*.
4. $\therefore V = \frac{1}{3}h(B + B' + \sqrt{BB'})$.	4. § 530, *b*.

601. Optional proof of § 600 by Cavalieri's Theorem.



1. Let the bases of cone C and pyramid P be equal, having area B; let their altitudes be equal to H. Let $A'B$ be a frustum of C with altitude h and upper base of area B'; let $E'F$ be a frustum of P also with altitude h. Let $A''B''$ be a section of frustum $A'B$ made by a plane parallel to AB at distance d from it. Let $E''F''$ be a section of frustum $E'F$ made by a plane parallel to EF and at distance d from it.

2. $\dfrac{\text{Area } A'B'}{\text{Area } AB} = \dfrac{(H-h)^2}{H^2}$; $\dfrac{\text{area } E'F'}{\text{area } EF} = \dfrac{(H-h)^2}{H^2}$. § 588, § 524

$\therefore$ Area $E'F'$ = area $A'B'$, § 254

$\dfrac{\text{Area } A''B''}{\text{Area } AB} = \dfrac{(H-d)^2}{H^2}$; $\dfrac{\text{area } E''F''}{\text{area } EF} = \dfrac{(H-d)^2}{H^2}$.

$\therefore$ area $A''B''$ = area $E''F''$.

3. $\therefore$ volume $A'B$ = volume $E'F$. § 532

4. But volume $E'F = \frac{1}{3}h(B + B' + \sqrt{BB'})$. § 539

5. $\therefore$ volume $A'B = \frac{1}{3}h(B + B' + \sqrt{BB'})$ Ax. 2

Note. Observe that C is not assumed to be a circular cone.

Ex. 41. Find the volume of a frustum of a circular cone whose altitude is 15 in., and whose bases have diameters of 8 in. and 6 in.

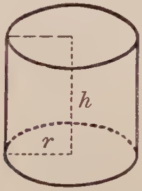
OPTIONAL TOPIC B

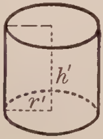
602. Similar cylinders of revolution are generated by the revolution of similar rectangles about corresponding sides.

Similar cones of revolution are generated by the revolution of similar right triangles about corresponding sides.

PROPOSITION XI. THEOREM

603. *The lateral or total areas of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*





Hypothesis. A and A' are the lateral areas, T and T' are the total areas, V and V' are the volumes, h and h' are the altitudes, and r and r' the radii of the bases, of two similar cylinders of revolution.

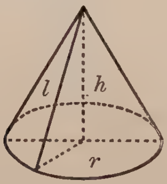
Conclusion. $\dfrac{A}{A'} = \dfrac{T}{T'} = \dfrac{h^2}{h'^2} = \dfrac{r^2}{r'^2}$; and $\dfrac{V}{V'} = \dfrac{h^3}{h'^3} = \dfrac{r^3}{r'^3}$.

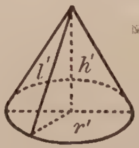
Proof:

	STATEMENTS	REASONS
1.	$\dfrac{h}{r} = \dfrac{h'}{r'}$.	**1.** Why?
2.	$\therefore \dfrac{h+r}{r} = \dfrac{h'+r'}{r'}$.	**2.** § 257.
3.	$\therefore \dfrac{h+r}{h'+r'} = \dfrac{r}{r'}$.	**3.** § 255.
4.	$\dfrac{A}{A'} = \dfrac{2\pi rh}{2\pi r'h'} = \dfrac{r}{r'} \times \dfrac{h}{h'} = \dfrac{r^2}{r'^2} = \dfrac{h^2}{h'^2}$.	**4.** § 573. Ax. 2.
5.	$\dfrac{T}{T'} = \dfrac{2\pi r(h+r)}{2\pi r'(h'+r')} = \dfrac{r}{r'} \times \dfrac{r}{r'} = \dfrac{r^2}{r'^2} = \dfrac{h^2}{h'^2}$.	**5.** § 573. Ax. 2.
6.	$\dfrac{V}{V'} = \dfrac{\pi r^2 h}{\pi r'^2 h'} = \dfrac{r^2}{r'^2} \times \dfrac{h}{h'} = \dfrac{r^2}{r'^2} \times \dfrac{r}{r'} = \dfrac{r^3}{r'^3}$ $= \dfrac{h^3}{h'^3}$.	**6.** § 575. Ax. 2.

PROPOSITION XII. THEOREM

604. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.*





Hypothesis. A and A' are the lateral areas, T and T' the total areas, V and V' are the volumes, s and s' are the slant heights, h and h' are the altitudes, and r and r' the radii of the bases, of two similar cones of revolution. (§ 602.)

Conclusion.
$$\frac{A}{A'} = \frac{T}{T'} = \frac{s^2}{s'^2} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2};$$
$$\text{and } \frac{V}{V'} = \frac{s^3}{s'^3} = \frac{h^3}{h'^3} = \frac{r^3}{r'^3}.$$

The proof is left to the pupil; model it after that of § 603.

Note. Recalling at this point the various theorems about the ratios of segments, of areas, and of volumes of similar figures, observe:

(*a*) That the ratio of any two corresponding segments of similar figures equals the ratio of any two other corresponding segments.

(*b*) That the ratio of the areas of any two corresponding surfaces of similar figures equals the ratio of the squares of any two corresponding segments; or equals the square of the ratio of any two corresponding segments.

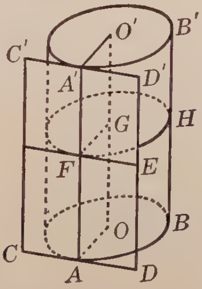
(*c*) That the ratio of the volumes of any two similar solids equals the ratio of the cubes of any two corresponding segments; or equals the cube of the ratio of any two corresponding segments.

OPTIONAL TOPIC C

605. A plane is tangent to a circular cylinder or to a circular cone when it contains one and only one element of the cylinder or of the cone.

PROPOSITION XIII. THEOREM

606. *A plane determined by an element of a circular cylinder and a tangent to the base at its extremity is tangent to the cylinder.*



Hypothesis. AA' is an element of circular cylinder AB'; line CD is tangent to the base AB at A; and plane CD' is determined by AA' and CD.

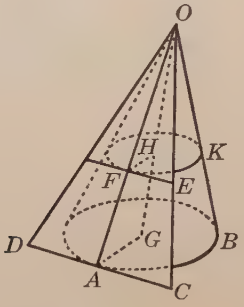
Conclusion. Plane CD' is tangent to the cylinder.

Proof:

STATEMENTS	REASONS
1. Let E be any point in plane CD', not in AA' and draw through E a plane $\parallel$ to the bases, intersecting CD' in line EF and the cylinder in circle FH.	1. § 564.
2. Draw axis OO'; then $OO' \parallel AA'$.	2. § 565.
3. The plane of OO' and AA' intersects the planes AB and FH in radii OA and GF.	3. Why?
4. $\angle GFE = \angle OAD =$ rt. $\angle$.	4. Prove it.
5. $\therefore FE \perp GF$, and tangent to $\odot FH$.	5. Why?
6. $\therefore$ point E lies outside the cylinder.	6. § 172, *b*.
7. $\therefore CD'$ is tangent to the cylinder.	7. § 605.

PROPOSITION XIV. THEOREM

607. *A plane determined by an element of a circular cone and a tangent to the base at its extremity is tangent to the cone.*



Hypothesis. OA is an element of circular cone OAB; line CD is tangent to base AB at A; and plane OCD is determined by OA and CD.

Conclusion. Plane OCD is tangent to the cone.

Proof:

STATEMENTS	REASONS
1. Let E be any point of plane ODC, not in OA, and draw through E a plane $\parallel$ base AB, intersecting ODC in line EF and the cone in $\odot FK$.	1. § 587.
2. Draw axis OHG. Then plane OAG cuts FHK in radius FH, and AGB in radius AG.	2. § 582.
3. $\angle HFE = \angle GAC$.	3. Prove it.
4. $\therefore FE \perp FH$, and FE is tangent to $\odot H$.	4. Prove it.
5. $\therefore E$ lies outside the cone.	5. Prove it.
6. $\therefore$ Every point of plane ODC, not in OA, lies outside the cone, or ODC is tangent to the cone.	6. Why?

BOOK IX

THE SPHERE

608. A **sphere** is a closed surface all points of which are equidistant from a point within, called the **center.**

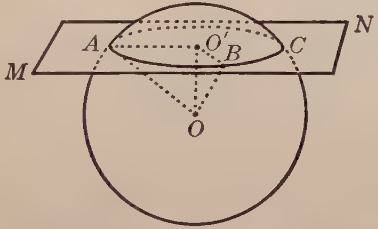
A **radius** of a sphere is the straight line segment drawn from its center to any point of it.

A **diameter** of a sphere is the straight line segment drawn through the center having its extremities in the sphere.

A point is *within*, *on*, or *outside* a sphere according as its distance from the center is less than, equal to, or greater than the radius.

*PROPOSITION I. THEOREM

609. *If a plane intersects a sphere, the intersection is a circle.*



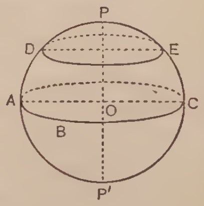
Hypothesis. ABC is the intersection of the sphere whose center is O and plane MN.

Conclusion. Intersection ABC is a ⊙.

Proof:

STATEMENTS	REASONS
1. Let OO', ⊥ plane MN, meet it at O'.	1. § 423.
2. Let A be a particular point and B any other point on ABC. Draw OA, OB, $O'A$, and $O'B$.	2. § 5, *a*.
3. Then $O'A = O'B$, and ABC is a circle.	3. Prove it.

610. A **great circle** of a sphere is the intersection of it and a plane passing through its center ; as ⊙ABC.



A **small circle** of a sphere is the intersection of it and a plane which does not pass through its center ; as ⊙DE.

The **axis** of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle.

The **poles of a circle of a sphere** are the extremities of the axis of the circle ; as P and P'.

611. Cor. 1. *The axis of a circle of a sphere passes through the center of the circle.*

612. Cor. 2. *All great circles of a sphere are equal.*

613. Cor. 3. *Every great circle bisects the sphere.*

For, if the portions of the interior of the sphere formed by the plane of the great circle be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two surfaces will coincide throughout.

614. Cor. 4. *The planes of any two great circles of a sphere intersect each other along a diameter of the sphere.*

615. Cor. 5. *Between any two points of a sphere, not the extremities of a diameter, one and only one arc of a great circle, less than a semicircle, can be drawn.*

For the two points, with the center of the sphere, determine a plane which intersects the sphere in the required arc.

616. Cor. 6. *Through any three points of a sphere, one and only one small circle can be drawn.*

For these three points determine a plane which intersects the sphere in one and only one circle through the three given points.

617. **Two spheres are congruent** when their radii are equal. All radii and diameters of the same sphere or congruent spheres are equal.

618. **A sphere may be generated** by the revolution of a semicircle about its diameter as an axis.

Ex. 1. How many great circles can be passed through two points which are the extremities of a diameter of a sphere?

Ex. 2. What kind of circle of the earth is the equator?

Ex. 3. Speaking strictly, is it accurate to speak of the North Pole of the earth? Of what circle or circles is it a pole?

Ex. 4. What kind of circles of the earth are the parallels of latitude?

Ex. 5. What kind of circles of the earth are the meridians?

Ex. 6. Into how many parts do two great circles of a sphere divide the sphere?

Ex. 7. Into how many parts do three great circles of a sphere divide the sphere, if they do not all have a common diameter?

Ex. 8. Prove that all circles of a sphere made by parallel planes have the same axis and the same poles.

Ex. 9. Given a point of a sphere. Prove that it is the pole of one and only one great circle.

Ex. 10. How many small circles of a sphere pass through two given points on the surface of the sphere, if the points are not the ends of a diameter?

Ex. 11. Prove that a great circle of a sphere which passes through one pole of another circle must pass through the other pole also.

Ex. 12. Prove that circles of a sphere made by planes equidistant from the center of the sphere are equal.

Ex. 13. State and prove the converse of Ex. 12.

Ex. 14. Prove that circles of a sphere made by planes unequally distant from the center of the sphere are unequal, the more remote being the smaller.

Ex. 15. State and prove the converse of Ex. 14.

Ex. 16. In how many points can two straight lines intersect? In how many points on one hemisphere can two great circles intersect?

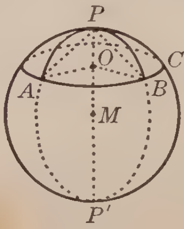
619. The great circles of a sphere in spherical geometry correspond to the straight lines of a plane in plane geometry. § 615 and Ex. 16 are two instances of this fact.

It is proved later that *the length of the arc of the great circle, less than a semicircle, between two points of a sphere is less than the length of any other curved line on the sphere between the two points.*

The **spherical distance** between two points of a sphere is defined to be the **arc of the great circle, less than a semicircle, drawn between them.**

PROPOSITION II. THEOREM

620. *The spherical distances of all points of a circle of a sphere from a pole of the circle are equal.*



Hypothesis. P and P' are the poles of $\odot ABC$ of sphere M.

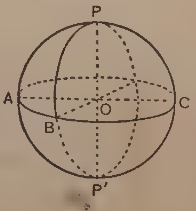
Conclusion. All points of $\odot ABC$ are equidistant from P, and also from P'.

Proof:

	STATEMENTS	REASONS
1.	From A and B, any two points of $\odot ABC$, draw great circle arcs PA and PB. Draw PA, PB.	1. § 615.
2.	Axis $PMP' \perp$ plane of $\odot ABC$ at center O.	2. § 611.
3.	Chord PA = chord PB.	3. Prove it.
4.	$\circledS PAP'$ and PBP' are equal.	4. § 612.
5.	$\therefore \overset{\frown}{PA} = \overset{\frown}{PB}$.	5. Why?
6.	$\therefore$ all points of $\odot ABC$ are equidistant from P. Also they are equidistant from P'.	6. Why?

621. The **polar distance of a circle of a sphere** is the spherical distance from the nearer of its poles to the circle, or from either pole if they are equally near.

622. Cor. *The polar distance of points of a great circle of a sphere from either pole is a quadrant.*



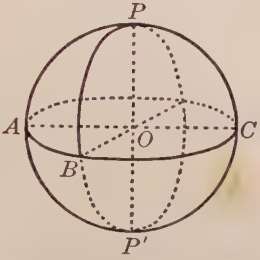
Note. The term quadrant in Spherical Geometry usually signifies a quadrant of a great circle.

Hypothesis. P is a pole of great $\odot ABC$; B is any point in $\odot ABC$, and PB is an arc of a great $\odot$.

Conclusion. Arc PB = a quadrant.

*PROPOSITION III. THEOREM

623. *A point of a sphere at a quadrant's distance from each of two points, not the extremities of a diameter of the sphere, is a pole of the great circle through those points.*



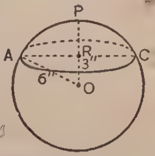
Hypothesis. P is on the sphere whose center is O. $\overset{\frown}{AB}$ is an arc of great $\odot ABC$, not a semicircle. $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ are quadrants of great ⓢ.

Conclusion. P is a pole of great $\odot ABC$.

Proof:	STATEMENTS	REASONS
1.	Draw PO, OA, and OB.	1. § 5, *a*.
2.	$PO \perp OA$, and $PO \perp OB$.	2. Prove it.
3.	$\therefore PO \perp$ plane ABC at O.	3. Why?
4.	$\therefore P$ is the pole of $\odot ABC$.	4. § 610.

Ex. 17. (*a*) The radius of a sphere is 6 in. What is the radius of the small circle made by a plane 3 in. from the center of the sphere?

(*b*) What is the length of $\overset{\frown}{AP}$, the polar distance of any point of the small circle of part (*a*)?



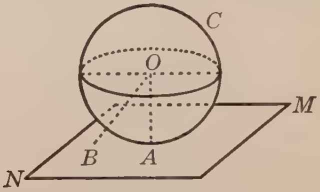
624. If a plane or a line has only one point in common with a sphere, it is said to be **tangent to the sphere.** The *sphere is said to be tangent to the plane or line.* The point common to the plane or line and the sphere is called the **point of contact or tangency.**

625. Two spheres are tangent to each other when each is tangent to the same plane at the same point.

Metcalf

PROPOSITION IV. THEOREM

626. *If a plane is perpendicular to a radius of a sphere at its outer extremity, it is tangent to the sphere.*



Hypothesis. Plane $MN \perp$ radius OA of sphere O at A.
Conclusion. Plane MN is tangent to the sphere.

Proof:

	Statements	Reasons
1.	Join O to any point B of MN, except A.	1. § 5, *a*.
2.	$OB > OA$.	2. Why?
3.	$\therefore$ B lies outside sphere O.	3. § 608.
4.	$\therefore$ MN is tangent to sphere O.	4. § 624.

627. Cor. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

OA is the shortest segment from O to MN. $\therefore OA \perp MN$.

Ex. 18. Prove that a straight line perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.

Ex. 19. How many straight lines can be tangent to a sphere at a point of the sphere?

Ex. 20. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

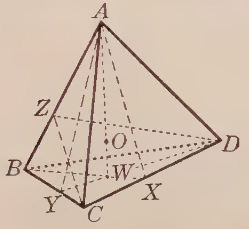
Ex. 21. Prove that all lines tangent to a sphere at a point of the sphere lie in the plane tangent to the sphere at that point.

Ex. 22. How many straight lines can be tangent to a sphere from a point outside the sphere? Compare the lengths of these tangents, from the point to the points of contact.

628. A sphere is **inscribed in a polyedron** if it is tangent to all the faces of the polyedron; and it is **circumscribed about the polyedron** if it passes through all the vertices of the polyedron.

PROPOSITION V. THEOREM

629. *A sphere can be inscribed in any tetraedron.*



Hypothesis. $ABCD$ is any tetraedron.

Conclusion. A sphere can be inscribed in $ABCD$.

Plan. Find a point equidistant from the faces of $ABCD$.

Proof:

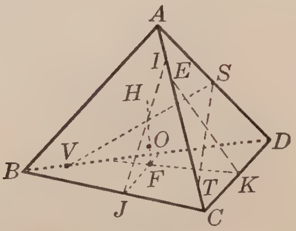
STATEMENTS	REASONS
1. Let plane ABX bisect diedral $\angle AB$, and plane ADY bisect diedral $\angle AD$. Then ABX intersects ADY in line AW.	1. § 413.
2. Every point on AW is equidistant from planes ABC and ABD; and also equidistant from planes ABD and ACD.	2. § 467.
3. ∴ every point on AW is equidistant from planes ABC, ABD, and ACD.	3. Ax. 1
4. Let CDZ, bisecting diedral $\angle CD$, intersect AW in point O. Then O is equidistant from BCD and ACD.	4. Why?
5. ∴ O is equidistant from all the faces of $ABCD$.	5. Step 3; Ax. 1.
6. ∴ the sphere with center O and radius equal to the distance from O to plane BCD will be tangent to each face of $ABCD$.	6. § 626.
7. ∴ sphere O is inscribed in $ABCD$.	7. § 628.

Ex. 23. Prove that the points of contact of all lines tangent to a sphere from an exterior point lie in a circle of the sphere.

Ex. 24. What figure is formed by the tangents in Ex. 22?

*PROPOSITION VI. THEOREM

630. *A sphere can be circumscribed about any tetraedron.*



Hypothesis. $ABCD$ is any tetraedron.

Conclusion. A sphere can be circumscribed about $ABCD$.

Proof:

Statements	Reasons
1. The locus of points equidistant from B and C is plane IJF, $\perp BC$ at its mid-point J. The locus of points equidistant from C and D is plane EKF, $\perp CD$ at its mid-point K.	1. § 465.
2. These planes intersect in $HF \perp BCD$.	2. § 460.
3. The locus of points equidistant from A and D is plane TSV, $\perp AD$ at its mid-point S, and intersecting HF at O.	3. Why?
4. O is equidistant from B and C, from C and D, and from A and D.	4. Why?
5. $\therefore$ O is equidistant from A, B, C, and D.	5. Why?
6. $\therefore$ a sphere with center O and radius OA will pass through A, B, C, and D.	6. § 608.

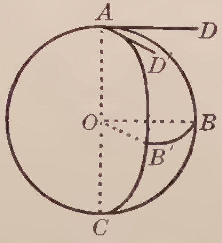
631. Cor. *A sphere is determined by four non-coplanar points, of which no three lie in a straight line.*

632. The measure of the **angle between two intersecting curves** is the measure of the angle formed by the tangents to the curves at the point of intersection.

A **spherical angle** is the angle between two intersecting arcs of great circles.

*PROPOSITION VII. THEOREM

633. *A spherical angle is measured by the arc of a great circle having the vertex of the angle as a pole and included between the sides of the angle, extended if necessary.*



Hypothesis. ABC and $AB'C$ are arcs of great ⓢ on the sphere O; lines AD and AD' are tangent to ABC and $AB'C$, respectively, and BB' is the arc of the great ⊙ having A as a pole, and included between arcs ABC and $AB'C$.

Conclusion. $\angle BAB'$ is measured by arc BB'.

Proof:

STATEMENTS	REASONS
1. Draw diameter AOC, and OB and OB'.	1. § 614.
2. Arcs AB and AB' are quadrants.	2. § 622.
3. $\therefore AO \perp OB$ and $AO \perp OB'$.	3. Prove it.
4. $AO \perp AD$; and $AO \perp AD'$.	4. Why?
5. $\therefore OB \parallel AD$ and $OB' \parallel AD'$.	5. Prove it.
6. $\therefore \angle DAD' = \angle BOB'$.	6. § 442.
7. $\therefore \angle DAD'$ is measured by arc BB'.	7. Prove it.
8. $\therefore \angle BAB'$ is measured by arc BB'.	8. § 632.

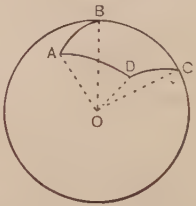
634. Cor. 1. *The angle between the arcs of two great circles has the same measure as the diedral angle formed by the planes of the circles.*

635. Cor. 2. *An arc of a great circle drawn to another great circle from the latter's pole is perpendicular to that great circle.* Prove $AB' \perp B'B$.

Ex. 25. How large is $\angle B'AB$, if $\angle B'OB$, is 40°?

SPHERICAL POLYGONS

636. A **spherical polygon** is a closed line on the surface of a sphere consisting of arcs of three or more great circles. We shall consider only convex spherical polygons. (See § 121.)



The arcs are the **sides** of the polygon.

Two consecutive sides form an **angle** and meet at a **vertex** of the polygon.

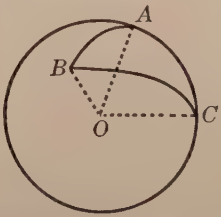
The definitions for the various kinds of triangles correspond exactly to those for plane triangles.

637. *The planes of the sides of a spherical polygon form a polyedral angle,* whose vertex is the center of the sphere, whose face angles are measured by the sides of the spherical polygon, and whose diedral angles have the same measure as the angles of the polygon.

Metcalf

PROPOSITION VIII. THEOREM

638. *Any side of a spherical triangle is less than the sum of the other two sides.*



Hypothesis. ABC is any spherical triangle on sphere O.

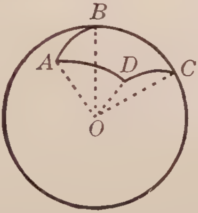
Conclusion. Any side < the sum of the other two sides.

Proof:

	STATEMENTS	REASONS
1.	Join A, B, and C to center O.	1. § 5, a.
2.	Then $\angle COA < \angle BOA + \angle BOC$.	2. § 475.
3.	But the measure of $\angle COA$ = the measure of $\overset{\frown}{AC}$; etc.	3. Why?
4.	$\therefore \overset{\frown}{AC} < \overset{\frown}{AB} + \overset{\frown}{BC}$.	4. Ax. 2.

PROPOSITION IX. THEOREM

639. *The sum of the sides of a convex spherical polygon is less than 360°.*



Hypothesis. $ABCD$ is a convex spherical polygon.

Conclusion. $\overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DA} < 360°$.

Proof:

STATEMENTS	REASONS
1. Form polyedral angle $O\text{–}ABCD$.	1. § 637.
2. $\angle AOB + \angle AOD + \angle DOC + \angle BOC < 360°$.	2. § 476.
3. $\angle AOB$ is measured by $\overset{\frown}{AB}$; etc.	3. Why?
4. $\therefore$ the sum of the measures of $\overset{\frown}{AB}$, $\overset{\frown}{AD}$, $\overset{\frown}{DC}$, and $\overset{\frown}{BC} < 360°$, or $\overset{\frown}{AB} + \overset{\frown}{AD} + \overset{\frown}{DC} + \overset{\frown}{BC} < 360°$ on a great ⊙.	4. Ax. 2,

640. Cor. *The sum of the sides of any convex spherical polygon is less than a great circle of the sphere.*

Note. Propositions VIII and IX and other theorems about spherical polygons are formulated from corresponding theorems about polyedral angles by replacing in the latter the words "face angle" and "diedral angle" by "side" and "angle" respectively.

Ex. 26. If two sides of a spherical triangle measure 80° and 70° respectively, the third side must be less than what arc?

Ex. 27. The perimeter of a convex spherical polygon on a sphere of radius 12 inches must be less than what length?

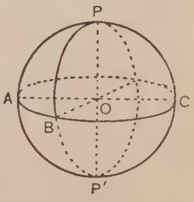
Ex. 28. A spherical angle intercepts an arc of 45° on the great circle whose pole is the vertex of the angle.

(*a*) How many degrees are there in the angle?

(*b*) How many degrees are there in the corresponding diedral angle?

POLAR TRIANGLES

641. Drawing great circles on a sphere of known radius. Using as radius AP, the chord of a quadrant, and a point P as center, *the resulting curve on the sphere* will be a great circle, and P is one of its poles. (Proved by § 623.)



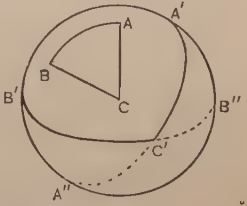
Observe that *the circle is drawn from one of its poles, and not from its center.*

This *radius* is the hypotenuse of the isosceles right triangle whose sides equal the radius of the sphere.

Ex. 29. What is the *approximate length*, correct to the nearest tenth, of the chord of a quadrant on a sphere of radius 10 in.?

642. The polar triangle of a given triangle is formed by drawing great circles from the vertices of the given triangle as poles. These circles divide the sphere into eight parts whose boundaries are triangles.

Thus, if circle $B'C'B''$ be drawn with vertex A of spherical $\triangle ABC$ as pole, circle $A'C'A''$ with B as pole, and circle $A'B'A''B''$ with C as pole, the sphere is divided into eight ⟁; namely, $\triangle A'B'C'$, $\triangle A'B''C'$, $\triangle A''B'C'$, and $\triangle A''B''C'$ on the hemisphere represented in the figure, and four others on the opposite hemisphere.



Of these eight spherical ⟁, one is the polar triangle of ABC, *and is determined as follows:*

Of the intersections, A' and A'' of circles drawn with B and C as poles, A', which lies on the same side of $B'C'B''$ as A, is selected as vertex of the polar triangle; and similarly for the other intersections.

Thus, $A'B'C'$ is the polar $\triangle$ of ABC.

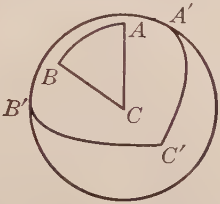
Ex. 30. If you have a spherical blackboard, locate a point P and draw the great circle of which it is the pole, as in § 641.

Ex. 31. (*a*) Place on the sphere two points. Locate the pole of the great circle which will pass through these points, by § 623.

(*b*) Using this point as pole, draw the great circle through the two given points.

*PROPOSITION X. THEOREM

643. *If one spherical triangle is the polar triangle of another, then the second is the polar of the first.*



Hypothesis. $\triangle A'B'C'$ is the polar $\triangle$ of $\triangle ABC$.

Conclusion. $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.

Proof:

	Statements	Reasons
1.	B is the pole of $\overset{\frown}{A'C'}$.	**1.** § 642.
2.	$\therefore$ $\overset{\frown}{BA'}$ is a quadrant.	**2.** § 622.
3.	C is the pole of $\overset{\frown}{A'B'}$.	**3.** Why?
4.	$\therefore$ $\overset{\frown}{CA'}$ is a quadrant.	**4.** Why?
5.	$\therefore$ A' is the pole of $\overset{\frown}{BC}$.	**5.** § 623.
	Similarly C' is the pole of $\overset{\frown}{AB}$, and B' of $\overset{\frown}{AC}$.	
6.	$\therefore$ $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.	**6.** § 642.

Note 1. Two spherical triangles, of which each is the polar of the other, are called *polar triangles.*

Note 2. An excellent class exercise, in case there is a spherical blackboard in the room, is to carry through the following construction:

(*a*) Locate on one hemisphere three points A, B, and C which are not on the same great circle.

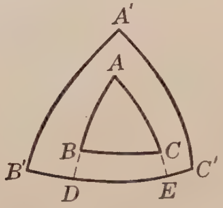
(*b*) Draw great circle arcs from these points as poles, forming $\triangle A'B'C'$, following § 641.

(*c*) From A', B', and C' as poles draw great circle arcs. These should intersect in A, B, and C.

(*d*) Then $\triangle ABC$ and $\triangle A'B'C'$ are polar triangles. Repeat or give the proof of § 643 from this figure.

*PROPOSITION XI. THEOREM

644. *In two polar triangles, each angle of either is the supplement of that side of the other of which it is pole.*



Hypothesis. $\triangle ABC$ and $A'B'C'$ are polar $\triangle$, point A being the pole of $\overset{\frown}{B'C'}$; etc.; also A' is the pole of $\overset{\frown}{BC}$, etc.

Conclusion. $\angle A + \overset{\frown}{B'C'} = 180°$; $\angle B + \overset{\frown}{A'C'} = 180°$; etc. $\angle A' + \overset{\frown}{BC} = 180°$; $\angle B' + \overset{\frown}{AC} = 180°$; $\angle C' + \overset{\frown}{AB} = 180°$.

Proof:

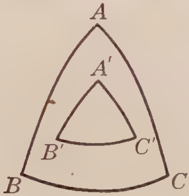
STATEMENTS	REASONS
1. Let $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ meet $\overset{\frown}{B'C'}$ at D and E.	**1.** Great ⓢ intersect.
2. B' is a pole of $\overset{\frown}{ACE}$. $\therefore \overset{\frown}{B'E} = 90°$.	**2.** Why?
3. Similarly, $\overset{\frown}{C'D} = 90°$.	**3.** § 622.
4. $\therefore \overset{\frown}{B'E} + \overset{\frown}{C'D} = 180°$.	**4.** Ax. 3.
5. $\therefore \overset{\frown}{B'C'} + \overset{\frown}{DE} = 180°$.	**5.** Ax. 7.
6. But $\angle A$ is measured by $\overset{\frown}{DE}$.	**6.** § 633.
7. $\therefore$ the sum of the measures of $\overset{\frown}{B'C'}$ and $\angle A = 180°$, or $\angle A + \overset{\frown}{B'C'} = 180°$. Similarly $\angle B + \overset{\frown}{A'C'} = 180°$, etc.	**7.** Ax. 2.

Ex. 32. If the sides of a spherical triangle are 77°, 123°, and 95°, how many degrees are there in each angle of its polar triangle?

Ex. 33. If the angles of a spherical triangle are 86°, 131°, and 68°, how many degrees are there in each side of its polar triangle?

*PROPOSITION XII. THEOREM

645. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*



Hypothesis. $\triangle ABC$ is any spherical $\triangle$.

Conclusion. $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

Proof:

	Statements	Reasons
1.	Let $\triangle A'B'C'$ be the polar $\triangle$ of $\triangle ABC$, A being the pole of $\overset{\frown}{B'C'}$, B of $\overset{\frown}{A'C'}$, etc.	1. § 642.
2.	$\angle A + \overset{\frown}{B'C'} = 180°$; $\angle B + \overset{\frown}{A'C'} = 180°$; etc.	2. § 644.
3.	$\angle A + \angle B + \angle C + (\overset{\frown}{B'C'} + \overset{\frown}{A'C'} + \overset{\frown}{A'B'}) = 540°$.	3. Ax. 3.
4.	But $\overset{\frown}{B'C'} + \overset{\frown}{A'C'} + \overset{\frown}{A'B'} < 360°$.	4. § 639.
5.	$\therefore \angle A + \angle B + \angle C > 180°$.	5. Ax. 13.
6.	Also $\overset{\frown}{B'C'} + \overset{\frown}{A'C'} + \overset{\frown}{A'B'} > 0°$.	6. Obvious.
7.	$\therefore \angle A + \angle B + \angle C < 540°$.	7. Ax. 13.

646. The **spherical excess** of a spherical triangle is the sum of its angles diminished by 180°.

The **spherical excess** of a **spherical** n**-gon** is the sum of its angles diminished by $(n - 2)$ st. $\measuredangle$.

Ex. 34. Prove that a spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.

647. A spherical triangle having two right angles is called a bi-rectangular triangle, and one having three right angles a tri-rectangular triangle.

Note. Additional Exercises 85–102, pp. 464–5, can be done now.

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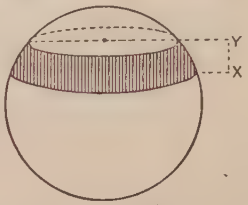
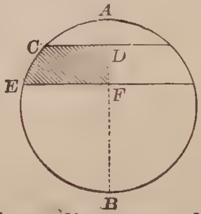
648. A **zone** is the portion of a sphere included between two parallel planes.



The circles which bound the zone are its **bases,** and the distance between their planes is its **altitude.**

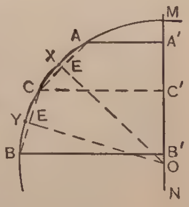
A zone of one base lies between one plane and a parallel tangent plane.

If semicircle $ACEB$ be revolved about diameter AB as an axis, and CD and EF are lines $\perp AB$, then arc CE generates a zone whose altitude is DF, and arc AC a zone whose altitude is AD.



649. Application of limits to zones.

Let O be the center of $\widehat{ACB}$, and MON be a diameter of the circle. Let AA' and BB' be $\perp OM$. Let C bisect $\widehat{AB}$, and draw broken line ACB. If $\widehat{AB}$ be revolved about OM as axis, it generates a zone, and segments AC and CB generate frustums of cones.



It is evident that the sum of the areas of the surfaces generated by AC and CB is less than the area of the zone generated by $\widehat{ACB}$.

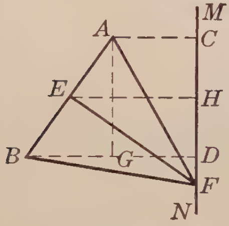
Assume $\widehat{AC}$ and $\widehat{CB}$ to be bisected at X and Y, and imagine the broken line $AXCYB$. It is evident that the area of the surface generated by broken line $AXCYB$ is greater than that generated by ACB but is still less than that of the zone generated by $\widehat{ACB}$.

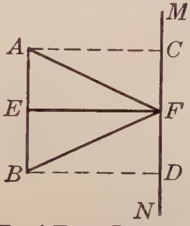
If $\widehat{AB}$ be successively halved, it is evident that the surfaces generated by the resulting broken line segments approaches as limit the zone generated by $\widehat{AB}$; also, as the chords like AC approach the limit zero, their distance from center O approaches the radius of $\widehat{AB}$ as limit.

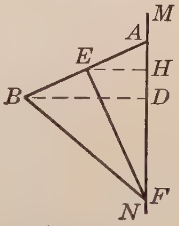
Ex. 35. Prove that the sum of the angles of a spherical hexagon is greater than 8, and less than 12, right angles.

PROPOSITION XIII. THEOREM

650. *If an isosceles triangle be revolved about an axis which passes through its vertex and lies in its plane but does not cross the triangle, the area of the surface generated by its base equals the projection of its base on the axis, multiplied by the circumference of the circle whose radius is the altitude to its base.*







Hypothesis. In $\triangle ABF$, $AF = BF$, and $EF \perp AB$.
MN lies in the plane of $\triangle ABF$, passing through F.
CD is the projection of AB on MN.
$\triangle ABF$ is revolved about MN as axis.

Conclusion. Area $AB^* = CD \times 2\,\pi EF$.

Proof on Fig. 1. STATEMENTS	REASONS
1. Draw $EH \perp CD$; draw $AG \perp BD$.	**1.** § 82.
2. AB generates the lateral surface of a frustum of a cone of revolution. $\therefore$ area $AB = AB \times 2\,\pi EH$.	**2.** § 599.
3. $\triangle ABG \sim \triangle EFH$.	**3.** Prove it.
4. $\therefore \dfrac{AB}{EF} = \dfrac{AG}{EH}$, and $AB \times EH = AG \times EF$.	**4.** Prove it.
5. $\therefore AB \times 2\,\pi EH = AG \times 2\,\pi EF$. Or $AB \times 2\,\pi EH = CD \times 2\,\pi EF$.	**5.** Why?
6. $\therefore$ area $AB = CD \times 2\,\pi EF$.	**6.** Ax. 2.

Ex. 36. What is the spherical excess of a triangle whose angles are 100°, 95°, and 65°, respectively?

Ex. 37. What is the spherical excess of a tri-rectangular triangle?

Ex. 38. Prove the theorem of § 650 by use of Fig. 2.

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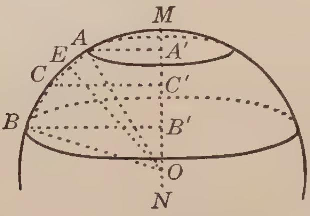
* Area AB denotes "the area of the surface generated by AB."

PROPOSITION XIV. THEOREM

651. *The area of a zone is equal to its altitude multiplied by the circumference of a great circle.*



Hypothesis. O is center of $\widehat{AB}$. $\widehat{AB}$ is revolved about MON as axis; AA' and $BB' \perp OM$; r is the radius of $\widehat{AB}$.

Conclusion. Area of zone generated by $AB = A'B' \times 2\,\pi r$.

Proof:

STATEMENTS	REASONS
1. Bisect AB at C; draw $CC' \perp OM$. Draw AC, and BC. Draw $OE \perp AC$.	1. All possible constructions.
2. If the figure be revolved about MO then area $AC = A'C' \times 2\,\pi OE$, and area $CB = C'B' \times 2\,\pi OE$.	2. § 650.
3. $\therefore$ Area $ACB = (A'C' + C'B') \times 2\,\pi OE$. Or area $ACB = A'B' \times 2\,\pi OE$.	3. Ax. 3.
4. If the parts of $\widehat{AB}$ be repeatedly halved and the process of Steps 1–3 be repeated, the area of the surface generated always equals $A'B' \times 2\pi \times (a \perp \text{like } OE)$.	4. Steps 1–3.
5. But the surface generated $\longrightarrow$ the zone AB, and $OE \longrightarrow r$, if $\widehat{AB}$ be divided infinitely often.	5. § 649.
6. $\therefore$ area of zone $AB = A'B' \times 2\,\pi r$.	6. § 530, *b*.

652. Cor. 1. *If Z denotes the area of a zone, h its altitude, and r the radius of the sphere,*

$$Z = 2\,\pi r h.$$

653. Cor. 2. *The area of a sphere equals the square of its radius multiplied by* 4π.

A sphere is a zone whose altitude equals $2r$. Letting A represent the area of the sphere,

$$A = 2\pi r \times 2r, \text{ or } A = 4\pi r^2.$$

654. Cor. 3. *The area of a sphere equals the area of four great circles of the sphere.*

655. Cor. 4. *The areas of two spheres have the same ratio as the squares of their radii or the squares of their diameters.*

Ex. 39. What is the area of the zone of altitude 5 on the sphere of radius 10?

Ex. 40. What is the area of a zone of one base on a sphere of radius 12 in., if the generating arc of the zone contains 60°?

Ex. 41. What is the area of the sphere whose radius is 8 in.?

Ex. 42. What is the area of the sphere whose diameter is 15 in.?

Ex. 43. What is the area of the sphere whose diameter is d in.?

Ex. 44. Prove that the areas of two zones on the same sphere are to each other as their altitudes.

Ex. 45. What zones of the earth are zones of one base?

Ex. 46. What part of the surface of the earth lies between the equator and the parallel of latitude 30° N.?

Ex. 47. Compare the surface of a sphere with the lateral surface of the right circular cylinder circumscribed about the sphere.

Ex. 48. Find the radius of a sphere whose surface is equal to the entire surface of a cylinder of revolution, whose altitude is 12, and radius of base is 3.

Ex. 49. Prove that the surface of a sphere is equal to two thirds the entire surface of the right circular cylinder circumscribed about it.

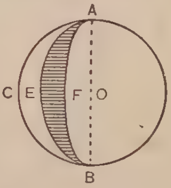
Ex. 50. What is the area of the sphere, one of the great circles of which has an area of 18π sq. in.?

Ex. 51. What is the area of the sphere whose radius is three times that of a second sphere, if the area of the second sphere is 20π sq. in.?

Ex. 52. What is the ratio of the areas of two spheres if the ratio of their radii is $\frac{1}{2}$? $\frac{1}{3}$? $\frac{2}{3}$?

MEASUREMENT OF SPHERICAL POLYGONS

656. A **lune** is the portion of a sphere bounded by two semicircles of great circles; as $BEAFB$.

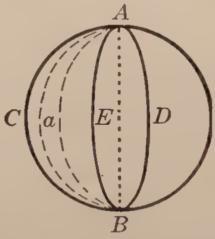


The **angle of a lune** is the angle between its bounding arcs, as $\angle EAF$.

657. *Two lunes on the same sphere or equal spheres are congruent if their angles are equal.*

PROPOSITION XV. THEOREM

658. *Two lunes on the same sphere or equal spheres have the same ratio as their angles.*



Hypothesis. $ACBD$ and $ACBE$ are lunes on sphere AB, having their $\measuredangle$ CAD and CAE commensurable.

Conclusion. $\dfrac{\text{Lune } ACBD}{\text{Lune } ACBE} = \dfrac{\angle CAD}{\angle CAE}.$

Proof:

	STATEMENTS	REASONS
1.	Let $\angle CAa$ be contained 5 times in $\angle CAD$ and 3 times in $\angle CAE$.	1. § 205.
2.	$\therefore \angle CAD : \angle CAE = 5 : 3.$	2. § 206.
3.	Lune $ACBD$ can be divided into 5 parts and lune $ACBE$ into 3 parts, all equal.	3. § 657.
4.	$\therefore \dfrac{\text{lune } ACBD}{\text{lune } ACBE} = \dfrac{5}{3}.$	4. § 206.
5.	$\therefore \dfrac{\text{lune } ACBD}{\text{lune } ACBE} = \dfrac{\angle CAD}{\angle CAE}.$	5. Why?

Note. The incommensurable case is given on page 451.

Ex. **53.** What figure is formed by two meridians of the earth?

659. A hemisphere may be considered a lune of angle 180° and a sphere, a lune of angle 360°.

660. Cor. 1. *A lune is to its sphere as the measure of its angle in degrees is to 360.*

661. Cor. 2. *If the radius of a sphere is r, and the angle of the lune, measured in degrees, is A, and the area of the lune is denoted by L_A,* then

$$L_A = \frac{A}{360} \times 4\,\pi r^2 \text{ or } \frac{\pi r^2 A}{90}.$$

Ex. 54. What part of a sphere is a lune of it whose angle is 45°?

Ex. 55. What part of the sphere is a lune of it whose angle is 60°?

Ex. 56. What is the area of a lune whose angle is 40° on the sphere whose radius is 15 in.?

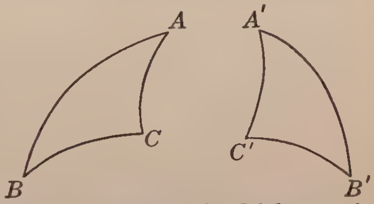
Ex. 57. What part of the surface of the earth is included between the 30th and 35th meridians?

Ex. 58. The area of a lune is 30. If the area of the sphere is 120, what is the angle of the lune?

SYMMETRIC SPHERICAL POLYGONS

662. Two spherical polygons, on the same or equal spheres, are **symmetric** when the sides and angles of one are equal, respectively, to the sides and angles of the other, *if the equal parts occur in opposite orders.*

If spherical $\triangle ABC$ and $A'B'C'$, on the same or equal spheres, have sides AB, BC, and CA equal, respectively, to sides $A'B'$, $B'C'$, and $C'A'$, and $\angle A$, B, and C to $\angle A'$, B', and C', and the equal parts occur in opposite orders, the $\triangle$ are symmetric.



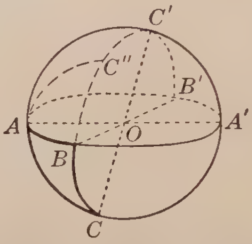
663. *Two symmetric spherical triangles, of which one is isosceles, are congruent, and therefore equal.*

Prove the triangles congruent by superposition.

664. **Diametrically opposite spherical triangles are two triangles whose corresponding vertices are at opposite ends of diameters.**

PROPOSITION XVI. THEOREM

665. *Two diametrically opposite spherical triangles are symmetric.*



Hypothesis. $\triangle ABC$ and $A'B'C'$ are so located on sphere O that AA', BB' and CC' are diameters.

Conclusion. $\triangle ABC$ is symmetric to $\triangle A'B'C'$.

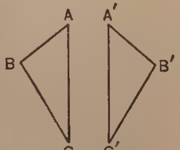
Proof:

STATEMENTS	REASONS
1. $\widehat{AB} = \widehat{A'B'}$; $\widehat{BC} = \widehat{B'C'}$; $\widehat{AC} = \widehat{A'C'}$.	1. Prove it.
2. $\angle ABOC = \angle A'B'OC'$; $\angle ACOB = \angle A'C'OB'$; etc.	2. § 453.
3. $\therefore \angle ABC = \angle A'B'C'$; $\angle ACB = \angle A'C'B'$; etc.	3. § 634.
4. The parts of $\triangle A'B'C'$ are in opposite order to those of $\triangle ABC$, for $\triangle A'B'C'$ can be placed in the position of $\triangle ABC''$; then, $A'B'C'$ is in counter-clockwise order and ABC in clockwise order.	4. § 62.
5. $\therefore \triangle ABC$ is symmetric to $\triangle A'B'C'$.	5. § 662.

Ex. 59. If $A'C'$ is a quadrant, how many degrees are there in AC?

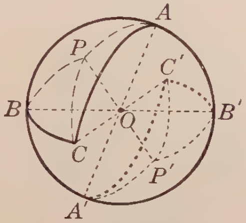
Ex. 60. Can $\triangle A'B'C'$ be made to coincide with $\triangle ABC$? Are these two triangles congruent?

Note. In plane geometry, we might have considered symmetric triangles. In the adjoining figure are two symmetric plane triangles; the corresponding parts are equal but are arranged in reverse order. These triangles also cannot be made to coincide *unless one is taken out of its plane and turned over.*



PROPOSITION XVII. THEOREM

666. *Two diametrically opposite spherical triangles are equal.*



Hypothesis. $\triangle$s ABC and $A'B'C'$ are such that AOA', BOB', and COC' are diameters of sphere O.

Conclusion. Area of $\triangle ABC$ = area of $\triangle A'B'C'$.

Proof:

	Statements	Reasons
1.	Let P be the pole of the small circle through A, B, and C; draw great circle arcs, PA, PB, and PC. Draw diameter POP', and great circle arcs $P'A'$, $P'B'$, and $P'C'$.	1. § 610; § 615.
2.	$\overset{\frown}{PA} = \overset{\frown}{PB}$, and $\therefore$ $\triangle PAB$ is isosceles.	2. Why?
3.	$\triangle PAB$ and $\triangle P'A'B'$ are symmetric.	3. § 665.
4.	$\therefore \triangle PAB \cong \triangle P'A'B'$. Similarly $\triangle PAC \cong \triangle P'A'C'$, $\triangle PBC \cong \triangle P'B'C'$.	4. § 663.
5.	$\therefore \triangle ABC = \triangle A'B'C'$.	5. Ax. 3.

667. Cor. *Two symmetric spherical triangles are equal.*

1. Let $\triangle XYZ$ be symmetric to $\triangle X'Y'Z'$. Let $\triangle X''Y''Z''$ be diametrically opposite $\triangle XYZ$. Then $\triangle X''Y''Z''$ is symmetric to $\triangle XYZ$ and equals $\triangle XYZ$.

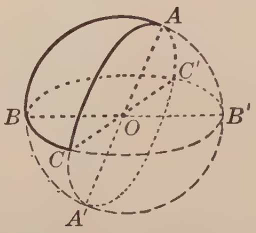
2. Since $\triangle X''Y''Z''$ and $\triangle X'Y'Z'$ are both symmetric to $\triangle XYZ$, their parts must be equal and arranged in the same order.

$$\therefore \triangle X''Y''Z'' \cong \triangle X'Y'Z'.$$

3. Since $\triangle X''Y''Z'' = \triangle XYZ$, $\therefore$ also $\triangle X'Y'Z' = \triangle XYZ$.

PROPOSITION XVIII. THEOREM

668. *A spherical triangle equals one half a lune of its sphere whose angle equals the spherical excess of the triangle.*



Hypothesis. $\triangle ABC$ is any spherical $\triangle$ on the sphere with center O. E represents the spherical excess of the triangle.

Conclusion. $\triangle ABC$ = one half a lune whose angle is E.

Proof:

STATEMENTS	REASONS
1. Complete ⓢ $ABA'B'$, $ACA'C'$, $BCB'C'$, and draw diameters AA', BB', and CC'.	1. § 614.
2. $\triangle ABC + \triangle ACB'$ = lune of $\angle B$.	2. Ax. 7.
3. $\triangle ABC + \triangle A'CB$ = lune of $\angle A$.	3. Ax. 7.
4. $\triangle ABC + \triangle AC'B$ = lune of $\angle C$.	4. Ax. 7.
5. But $\triangle AC'B = \triangle A'CB'$.	5. § 666.
6. $\therefore$ $\triangle ABC + \triangle A'CB'$ = lune of $\angle C$.	6. Ax. 2.
7. From Steps 2, 3, and 6, $2\triangle ABC + (\triangle ABC + \triangle ACB' + \triangle A'CB + \triangle A'CB')$ = lune of $\angle A$ + lune of $\angle B$ + lune of $\angle C$.	7. Ax. 3.
8. $\triangle ABC + \triangle ACB' + \triangle A'CB + \triangle A'CB'$ = lune 180°.	8. Ax. 7.
9. $\therefore$ $2\triangle ABC$ = lune $\angle A$ + lune $\angle B$ + lune $\angle C$ − lune 180°.	9. Ax. 2.
10. $\therefore$ $2\triangle ABC$ = lune of $(\angle A + \angle B + \angle C - 180°)$.	10. See note.
11. $\therefore$ $\triangle ABC = \frac{1}{2}$ lune of $\angle E$.	11. § 646.

Note. A lune of $\angle A$ + a lune of $\angle B$ = a lune of $\angle(A + B)$. Similarly a lune of $\angle A$ − a lune of $\angle B$ = a lune of $\angle(A - B)$.

669. Cor. 1. *If the radius of a sphere is r and the spherical excess of $\triangle ABC$ in degrees is E, then, by* § 668 and § 661,

$$\textit{area of } \triangle ABC = \tfrac{1}{2} L_E = \frac{1}{2}\frac{\pi r^2 E}{90} = \frac{\pi r^2 E}{180}.$$

670. Cor. 2. *The area of any spherical polygon whose excess is E is* $\dfrac{\pi r^2 E}{180}$.

If the polygon be divided into triangles by drawing diagonals from one vertex, the excess of the polygon equals the sum of the excesses of the triangles. Express the area of each triangle.

Ex. 61. Determine the area of the spherical triangle whose angles are 125°, 130°, and 105°, on a sphere whose radius is 10 in.

Suggestion. What is the excess of the $\triangle$? (§ 646.)

Ex. 62. What is the ratio of the areas of two spherical triangles on the same sphere, whose angles are 95°, 135°, and 140°, and 85°, 105°, and 120°, respectively?

Ex. 63. Determine the area of the spherical triangle whose angles are 103°, 112°, and 125°, on the sphere whose radius is 12 in.

Ex. 64. What is the area of the spherical hexagon whose angles are 120°, 140°, 148°, 155°, 162°, and 165° on the sphere whose radius is 15 in.?

Suggestion. What is the excess of a polygon? (§ 646.)

Ex. 65. The sides of a spherical triangle on the sphere whose radius is 15 in. are 40°, 65°, and 95°. Find the area of its polar triangle.

Ex. 66. Determine the part of the sphere intercepted by a triedral angle whose diedral angles are 85°, 55°, and 100°.

Ex. 67. What is the area of the spherical pentagon whose angles are 112°, 132°, 138°, 168°, and 150°, if the radius of the sphere is 20 in.?

Ex. 68. What part of the sphere is a trirectangular triangle?

Ex. 69. Compare the area of a trirectangular triangle of a sphere of radius 10 in. with the area of the plane triangle formed by the chords of the spherical triangle.

Ex. 70. A spherical degree is a birectangular spherical triangle whose third angle is 1°. What part of the sphere is included by it?

Ex. 71. What is the area of the birectangular spherical triangle whose third angle is 15°, on the sphere of area 66 sq. in.?

VOLUME OF A SPHERE

671. If a semicircle be revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a **spherical sector.**

If semicircle *ADB* in Fig. 1 be revolved about *AB* as axis, sector *OCD* generates a spherical sector.

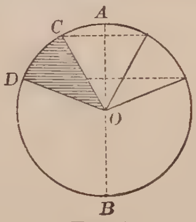


FIG. 1.

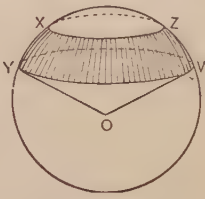
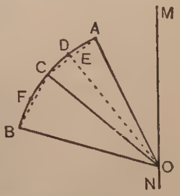


FIG. 2.

The zone generated by the arc of the circular sector is called the base of the spherical sector; as zone *XW*, in Fig. 2.

Note. "Vol. *OCD*" will denote the volume of the solid generated by revolving the portion of the plane within *OCD* around some axis.

672. Application of limits to spherical sectors. Let *O* be the center of arc *AB*, and *MON* be a diameter of the circle whose center is *O*. Let *C* bisect $\overset{\frown}{AB}$ and draw radii *OA*, *OB*, and *OC*. Draw *OE* ⊥ *AC*, and broken line *ACB*.



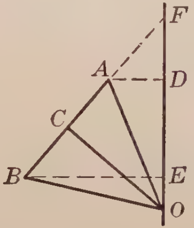
If sector *OAB* be revolved about *MON* as axis, it generates a spherical sector. The portion of the plane bounded by polygon *OACB* generates a solid which is less than the spherical sector generated by circular sector *OAB*.

Let arcs *AC* and *CB* be bisected at *D* and *F* respectively, and broken line *ADCFB* be drawn. When the figure is revolved about *MON*, the part of the plane bounded by the polygon *OADCFB* generates a solid more nearly equal to the spherical sector. If the process of halving the arcs be continued indefinitely, it is evident that the solid generated by the part of the plane bounded by the polygon *OA* ⋯ *C* ⋯ *B* approaches the spherical sector as limit.

The surface generated by broken line *A* ⋯ *C* ⋯ *B* approaches as limit the zone generated by the arc *ACB*.

PROPOSITION XIX. THEOREM

673. *If an isosceles triangle (and the surface inclosed by it) be revolved about a straight line in its plane which passes through its vertex but does not cross the interior of the triangle, the volume of the solid generated equals the area of the surface generated by its base multiplied by one third its altitude.*



Hypothesis. Isosceles $\triangle OAB$ and its interior are revolved about OF which lies in its plane. $OC \perp AB$.

Conclusion. Vol. OAB* $=$ area $AB \times \frac{1}{3} OC$.

Proof:

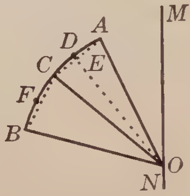
Statements	Reasons
1. Draw $AD \perp OF$ and $BE \perp OF$; let AB meet OF at F.	1. § 82.
2. Vol. $OBF =$ vol. $OBE +$ vol. BEF.	2. Ax. 7.
3. $\therefore$ vol. $OBF = \frac{1}{3}\pi \overline{BE}^2 \times OE + \frac{1}{3}\pi \overline{BE}^2 \times EF$.	3. § 595.
4. $\therefore$ vol. $OBF = \frac{1}{3}\pi \overline{BE}^2(OE + EF)$.	4. Why?
5. $\therefore$ vol. $OBF = \frac{1}{3}\pi BE \times BE \times OF$.	5. Ax. 7.
6. $\triangle BEF \sim \triangle OCF$; $\therefore BE \times OF = OC \times BF$.	6. Prove it, using § 286.
7. $\therefore$ vol. $OBF = \frac{1}{3}\pi BE \times OC \times BF$.	7. Why?
8. $\pi \times BE \times BF =$ area generated by FB.	8. § 592.
9. $\therefore$ vol. $OBF = \frac{1}{3} OC \times$ area BF. Similarly vol. $OAF = \frac{1}{3} OC \times$ area AF.	9. Ax. 2.
10. $\therefore$ vol. $OAB = \frac{1}{3} OC$ (area $BF -$ area AF).	10. Ax. 4.
11. $\therefore$ vol. $OAB = \frac{1}{3} OC \times$ area AB.	11. Ax. 2.

---

* See Note § 671, and Foot-note, p. 431.

PROPOSITION XX. THEOREM

674. *The volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one third the radius of the sphere.*



Hypothesis. Sector OAB of $\odot O$ is revolved about diameter MON as an axis; r is the radius of the sphere.

Conclusion. Vol. of the spherical sector generated by circular sector OAB = area of zone generated by $\widehat{AB} \times \frac{1}{3} r$.

Proof:

STATEMENTS	REASONS
1. Let C bisect $\widehat{AB}$. Draw AC, CB, OC, and $OE \perp AC$.	1. § 82.
2. Vol. $OAC = \frac{1}{3} OE \times$ area AC.	2. § 673.
3. Vol. $OCB = \frac{1}{3} OE \times$ area CB.	3. Why?
4. $\therefore$ vol. $OACB = \frac{1}{3} OE$ (area AC + area CB).	4. Ax. 3.
5. $\therefore$ vol. $OACB = \frac{1}{3} OE \times$ area ACB.	5. Ax. 7.
6. Repeatedly bisect the subdivisions of $\widehat{ACB}$. Then vol. $OA \cdot\cdot C \cdot\cdot B \longrightarrow$ vol. generated by sector OAB; area $A \cdot\cdot C \cdot\cdot B \longrightarrow$ area of zone generated by arc AB; and $OE \longrightarrow r$.	6. § 672.
7. $\therefore$ vol. generated by sector OAB = area of zone generated by $\widehat{AB} \times \frac{1}{3} r$.	7. § 530: a; b.

675. Cor. 1. *The volume of a sphere equals* $\frac{1}{3}$ *its radius multiplied by its area;* or, if the radius is r, and volume is V,

$$V = \tfrac{4}{3} \pi r^3.$$

For the sphere is a sector whose base is the surface of the sphere.

$$\therefore V = \tfrac{1}{3} r \times 4 \pi r^2, \text{ or } V = \tfrac{4}{3} \pi r^3.$$

676. Cor. 2. *If V denotes the volume of a spherical sector, h the altitude of the zone which forms its base, and r the radius of the sphere,*

$$V = 2\pi rh \times \tfrac{1}{3} r \text{ or } V = \tfrac{2}{3}\pi r^2 h. \qquad \S\ 652$$

677. Cor. 3. *The volumes of two spheres have the same ratio as the cubes of their radii.*

Ex. 72. Find the volume of the sphere whose radius is 9 in.

Ex. 73. Determine the volume of the metal in a spherical shell 12 in. in diameter and 1 in. thick.

Ex. 74. A spherical ball 9 in. in diameter is lowered into a cubical box filled with water, whose depth is 9 in. How many cubic inches of water will be left in the box?

Ex. 75. If a sphere 6 in. in diameter weights 351 oz., what is the weight of a sphere of the same material whose diameter is 8 in.?

Ex. 76. The outer diameter of a spherical shell is 10 in., and its thickness is 1 in. What is the weight, if a cubic inch of the metal weights one third pound?

Ex. 77. Find the area of the surface and the volume of the sphere inscribed in a cube whose edge is 8 in.

Ex. 78. Find the radius and the volume of a sphere, the area of whose surface is 256π sq. in.

Ex. 79. Prove that the volume of a sphere is two thirds the volume of its circumscribed cylinder.

Ex. 80. Within a sphere of radius r is inscribed a right circular cylinder whose altitude equals the diameter of its base. Compare its lateral area and volume with the area and volume of the sphere.

Ex. 81. Given a spherical surface of radius r and its circumscribed right circular cylinder. From the center of the sphere, draw lines to the points of the circles bounding the bases of the cylinder, thus forming two right circular cones. Compare the volume of the sphere with the difference between the volume of the cylinder and the sum of the volumes of the two cones.

Ex. 82. A cylindrical vessel, 8 in. in diameter, is filled with water. A ball is immersed in it, displacing water to the depth of $2\frac{1}{4}$ in. Find the diameter of the ball.

Ex. 83. What is the volume of the cube inscribed in a sphere of radius r?

Note. Supplementary Exercises 103–114, pp. 465–6, can be studied now.

OPTIONAL TOPICS

Topic A. General Theorems of Spherical Geometry.
Topic B. Incommensurable Case of §658.
Topic C. Spherical Segments, Pyramids, and Wedges.
Topic D. The shortest arc between two points.

TOPIC A

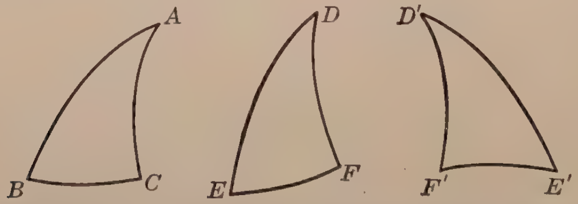
GENERAL THEOREMS OF SPHERICAL GEOMETRY

PROPOSITION XXI. THEOREM

678. *If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of one equal respectively to two sides and the included angle of the other:*

I. They are congruent if the equal parts occur in the same order.

II. They are symmetric if the equal parts occur in opposite orders.



I. Hypothesis. $\triangle ABC$ and $\triangle DEF$ are spherical $\triangle$ on the same sphere, or equal spheres, having $\overset{\frown}{AB} = \overset{\frown}{DE}$, $\overset{\frown}{AC} = \overset{\frown}{DF}$, and $\angle A = \angle D$; and the equal parts occur in the same order.

Conclusion. $\triangle ABC \cong DEF$.

Plan. Prove it by superposition as in § 62.

Ex. 84. Prove that the arc of a great circle bisecting the vertex angle of an isosceles spherical triangle is perpendicular to the base and bisects the base.

Ex. 85. If a great circle is perpendicular to and bisects a great circle arc, any point on it is equidistant from the ends of that arc.

II. Hypothesis. ABC and $D'E'F'$ are spherical ⟁ on the same sphere, or equal spheres, having $\overset{\frown}{AB} = \overset{\frown}{D'E'}$, $\overset{\frown}{AC} = \overset{\frown}{D'F'}$, and $\angle A = \angle D'$; *and the equal parts occur in opposite orders.*

Conclusion. ⟁ ABC and $D'E'F'$ are symmetric.

Proof. 1. Let $\triangle DEF$ be a spherical $\triangle$ on the same sphere, or an equal sphere, symmetric to $\triangle D'E'F'$, having

$$\overset{\frown}{DE} = \overset{\frown}{D'E'},\ \overset{\frown}{DF} = \overset{\frown}{D'F'},\ \text{and}\ \angle D = \angle D',$$

the equal parts occurring in opposite orders.

2. Then in spherical ⟁ ABC and DEF,

$$\overset{\frown}{AB} = \overset{\frown}{DE},\ \overset{\frown}{AC} = \overset{\frown}{DF},\ \text{and}\ \angle A = \angle D;$$

and the equal parts occur in the same order.

3. $\therefore \triangle ABC \cong \triangle DEF.$

4. $\therefore \triangle ABC$ is symmetric to $\triangle D'E'F'$.

PROPOSITION XXII. THEOREM

679. *If two spherical triangles on the same sphere, or on equal spheres, have a side and two adjacent angles of one equal respectively to a side and two adjacent angles of the other:*

I. They are congruent if the equal parts occur in the same order.

II. They are symmetric if the equal parts occur in opposite orders.

Part I. Prove it by superposition.

Part II. Model the proof after Part II of § 678.

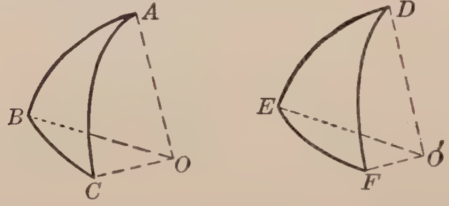
Ex. 86. Prove that the angles opposite the equal sides of an isosceles spherical triangle are equal.

Ex. 87. If great circle arcs, bisecting the base angles of an isosceles spherical triangle, are terminated by the opposite sides, these arcs are equal.

Ex. 88. If great circle arcs are drawn from the ends of the base of an isosceles spherical triangle to the mid points of the opposite sides, they are equal.

PROPOSITION XXIII. THEOREM

680. *If two spherical triangles on the same sphere, or on equal spheres, are mutually equilateral, they are mutually equiangular.*



Hypothesis. $\triangle ABC$ and $\triangle DEF$ are mutually equilateral spherical $\triangle$ on equal spheres O and O'. A, B, and C correspond to D, E, and F.

Conclusion. $\triangle ABC$ and $\triangle DEF$ are mutually equiangular.

Plan. 1. Draw the radii to A, B, C, D, E, and F.

2. Compare the face $\angle$ of triedral $\angle O\text{–}ABC$ and $O'\text{–}DEF$.

3. Compare the diedral $\angle$ of $O\text{–}ABC$ and $O'\text{–}DEF$ § 477.

4. Compare $\angle A$ and D, $\angle B$ and E, and $\angle C$ and F § 634.

681. Cor. *If two spherical triangles on the same sphere, or on equal spheres, are mutually equilateral:*

I. They are congruent if the equal parts occur in the same order.

II. They are symmetric if the equal parts occur in opposite orders.

Part I can be proved by superposition.

Part II follows at once from § 680 and § 662.

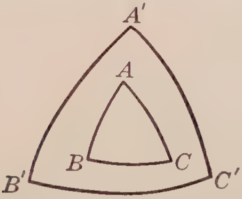
Ex. 89. Prove that the arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base is perpendicular to the base, and bisects the vertex angle.

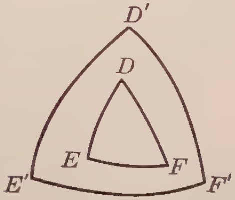
Ex. 90. Prove that the angle between the arcs of two great circles can be bisected.

Ex. 91. Prove that two points which are equidistant from the ends of an arc of a great circle, which is not a semicircle, determine an arc of a great circle which is perpendicular to and bisects the given arc.

PROPOSITION XXIV. THEOREM

682. *If two spherical triangles on the same sphere, or on equal spheres, are mutually equiangular, they are mutually equilateral.*





Hypothesis. $\triangle ABC$ and $\triangle DEF$ are mutually equiangular spherical $\triangle$ on the same sphere or equal spheres.

Conclusion. $\triangle ABC$ and $\triangle DEF$ are mutually equilateral.

Proof:

STATEMENTS	REASONS
1. Let $\triangle A'B'C'$ be the polar $\triangle$ of $\triangle ABC$, and $\triangle D'E'F'$ be the polar $\triangle$ of $\triangle DEF$.	1. § 642.
2. $\therefore$ $\triangle ABC$ is the polar of $\triangle A'B'C'$ and $\triangle DEF$ of $\triangle D'E'F'$.	2. § 643.
3. $\widehat{B'C'} + \angle A = 180°$; and $\widehat{E'F'} + \angle D = 180°$.	3. § 644.
4. $\therefore \widehat{B'C'} = \widehat{E'F'}$. Also $\widehat{A'B'} = \widehat{D'E'}$; etc.	4. Why?
5. $\therefore \angle A' = \angle D'$, $\angle B' = \angle E'$, and $\angle C' = \angle F'$.	5. § 680.
6. $\widehat{BC} + \angle A' = 180°$, and $\widehat{EF} + \angle D' = 180°$.	6. Why?
7. $\therefore \widehat{BC} = \widehat{EF}$. Also $\widehat{AC} = \widehat{DF}$; etc.	7. Why?

683. Cor. *If two spherical triangles on the same sphere or on equal spheres are mutually equiangular:*

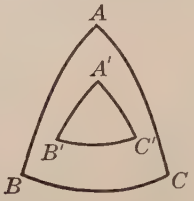
I. They are congruent if the equal parts are arranged in the same order. (Proved by § 682, and § 678.)

II. They are symmetric if the equal parts are arranged in opposite orders. (Proved by Hyp., § 682, and § 662.)

✓ **Ex. 92.** Are two planes triangles necessarily mutually equilateral when they are mutually equiangular?

PROPOSITION XXV. THEOREM

684. *If two angles of a spherical triangle are equal, the sides opposite are equal.*



Hypothesis. In spherical $\triangle ABC$, $\angle B = \angle C$.

Conclusion. $\widehat{AB} = \widehat{AC}$.

Proof:

STATEMENTS	REASONS
1. Let $\triangle A'B'C'$ be the polar $\triangle$ of $\triangle ABC$. Then also $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.	1. § 643.
2. $\widehat{A'C'} + \angle B = 180°$ and $\widehat{A'B'} + \angle C = 180°$.	2. Why?
3. $\therefore \widehat{A'C'} = \widehat{A'B'}$.	3. Why?
4. $\therefore \angle B' = \angle C'$.	4. Ex. 86, p. 445.
5. $\angle B' + \widehat{AC} = 180°$ and $\angle C' + \widehat{AB} = 180°$.	5. Why?
6. $\therefore \widehat{AC} = \widehat{AB}$.	6. Why?

Ex. 93. Prove that a point of a sphere which is equidistant from the ends of a great circle arc, lies on the second great circle arc, perpendicular to and bisecting the first one.

✓ **Ex. 94.** What is the locus of points on a sphere which are equidistant from the extremities of an arc of a great circle of that sphere, if that arc is not a semicircle?

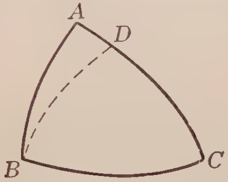
Ex. 95. Prove that the great circle arcs perpendicular to and bisecting the sides of a spherical triangle intersect in a point which is equidistant from the vertices of the triangle.

Ex. 96. On a sphere, an arc can be constructed making a given angle with a given arc at a point of the arc.

Ex. 97. On a sphere, an arc of a great circle can be drawn from a point, perpendicular to a given great circle arc.

PROPOSITION XXVI. THEOREM

685. *If two angles of a spherical triangle are unequal, the sides opposite them are unequal, the side opposite the greater angle being the greater.*



Hypothesis. In spherical $\triangle ABC$, $\angle ABC > \angle C$.

Conclusion. $\overset{\frown}{AC} > \overset{\frown}{AB}$.

Plan. Let $\overset{\frown}{BD}$ be a great ⊙ arc, meeting $\overset{\frown}{AC}$ at D, and making $\angle CBD = \angle C$. (Ex. 96, p. 448.)

Then prove: $\overset{\frown}{BD} = \overset{\frown}{CD}$; $\overset{\frown}{AD} + \overset{\frown}{BD} > \overset{\frown}{AB}$.

Substitute $\overset{\frown}{DC}$ for $\overset{\frown}{BD}$.

PROPOSITION XXVII. THEOREM

686. *If two sides of a spherical triangle are unequal, the angles opposite are unequal, the angle opposite the greater side being the greater.*

Hypothesis. $\overset{\frown}{AB} < \overset{\frown}{AC}$, in spherical $\triangle ABC$.

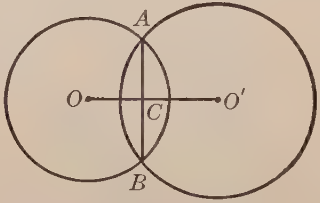
Conclusion. $\angle C < \angle B$.

Proof:

	STATEMENTS	REASONS
1.	If $\angle C = \angle B$; then $\overset{\frown}{AB} = \overset{\frown}{AC}$.	1. Why?
2.	But $\overset{\frown}{AB} < \overset{\frown}{AC}$.	2. Why?
3.	$\therefore \angle C$ is not $=$ to $\angle B$.	3. § 93.
4.	If $\angle C > \angle B$, then $\overset{\frown}{AB} > \overset{\frown}{AC}$.	4. Why?
5.	But $\overset{\frown}{AB} < \overset{\frown}{AC}$.	5. Why?
6.	$\therefore \angle C$ is not $> \angle B$.	6. Why?
7.	$\therefore \angle C < \angle B$.	

PROPOSITION XXVIII. THEOREM

687. *The intersection of two spheres is a circle, whose center is in a straight line joining the centers of the spheres and whose plane is perpendicular to that line.*



Hypothesis. O and O' are the centers of two intersecting spheres.

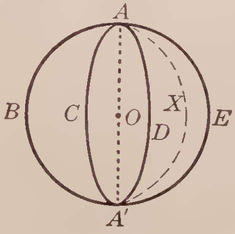
Conclusion. Spheres O and O' intersect in a $\odot$ whose center is on OO', and whose plane is $\perp$ OO'.

Proof : STATEMENTS	REASONS
1. Pass a plane through O, O', and any point A of the intersection. This plane intersects the spheres in two intersecting great circles. Let AB be the common chord of the circles, intersecting OO' at C.	1. § 610.
2. $\therefore OO' \perp AB$ and OO' bisects AB.	2. § 235.
3. Revolve the circles about OO' as axis; they generate their respective spheres, and A describes a curve on each sphere.	3. § 618.
4. As A revolves, AC, and $\therefore A$, always is in the plane $\perp OO'$ at C.	4. § 422.
5. At all positions of A, its distance from C is AC. $\therefore A$ describes a circle.	5. § 16.
6. No point outside $\odot ACB$ can lie in both surfaces; for, if there were such a point, the two surfaces would necessarily coincide.	6. § 631.
7. $\therefore$ the intersection of the spheres is a $\odot$.	7. § 412.

OPTIONAL TOPIC B

688. The incommensurable case of § 658.

Two lunes on the same sphere or equal spheres have the same ratio as their angles, when the angles are commensurable.



Hypothesis. Lunes $ABA'C$ and $ADA'E$ on sphere O, have $\angle BAC$ incommensurable with $\angle DAE$.

Conclusion. $\dfrac{\text{Lune } ABA'C}{\text{Lune } ADA'E} = \dfrac{\angle BAC}{\angle DAE}$.

Proof:

STATEMENTS	REASONS
1. Divide $\angle BAC$ into any number of equal parts, and apply one to $\angle DAE$ as unit of measure. Since $\angle DAE$ and $\angle BAC$ are incommensurable, this unit will not be contained exactly in $\angle DAE$.	1. § 205.
2. Suppose the unit is contained exactly in $\angle DAX$, and that $\angle XAE <$ the unit. Then $\dfrac{\text{lune } ABA'C}{\text{lune } ADA'X} = \dfrac{\angle BAC}{\angle DAX}$.	2. § 658.
3. Now let the number of subdivisions of $\angle BAC$ be increased infinitely. The length of the unit $\longrightarrow 0$.	3. § 529.
4. Since $\angle XAE <$ the unit, $\angle XAE \longrightarrow 0$, $\angle DAX \longrightarrow \angle DAE$, and lune $ADA'X \longrightarrow$ lune $ADA'E$.	4. § 529.
5. $\therefore \dfrac{\text{lune } ABA'C}{\text{lune } ADA'E} = \dfrac{\angle BAC}{\angle DAE}$.	5. § 530, *a*. § 530, *b*.

OPTIONAL TOPIC C

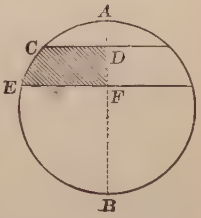
SPHERICAL SEGMENTS, PYRAMIDS, AND WEDGES

689. A **spherical segment** is the portion of the interior of a sphere included between two parallel planes which intersect the sphere.

The portions of the planes bounding the segment are the *bases* of the segment; the perpendicular between the planes is the *altitude* of the segment.

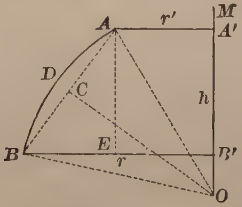
A spherical segment of one base is the spherical segment one of whose bounding planes is tangent to the sphere.

If a semicircle $ACEB$ be revolved about diameter AB as an axis, and CD and EF are perpendicular to AB, the portion of the plane bounded by $FECD$ generates a spherical segment whose altitude is DF, and whose bases have radii CD and EF respectively; the portion ACD generates a spherical segment of one base whose altitude is AD.



690. *If r and r' are the radii of the bases, h the altitude, and v the volume of a spherical segment, then*

$$v = \tfrac{1}{6}\pi h\{3(r^2 + r'^2) + h^2\}$$



Let O be the center of $\overset{\frown}{ADB}$; let AA' and BB' be $\perp$ to the diameter OM; let $AA' = r'$, $BB' = r$, $A'B' = h$. Let the whole figure revolve about OM as an axis.

Solution. 1. Draw OA, OB, and AB; draw $OC \perp AB$, and $AE \perp BB'$. Let $OA = R$.

2. Now, vol. $ADBB'A' =$ vol. $ACBD +$ vol. $ABB'A'$.
3. Also, vol. $ACBD =$ vol. $OADB -$ vol. OAB.
4. vol. $OADB = \frac{2}{3}\pi R^2 h$. § 676
5. And, vol. $OAB =$ area $AB \times \frac{1}{3} OC$ § 673
6. $= h \times 2\pi OC \times \frac{1}{3} OC$ § 650
7. $= \frac{2}{3}\pi \overline{OC}^2 h$.
8. $\therefore$ vol. $ACDB = \frac{2}{3}\pi R^2 h - \frac{2}{3}\pi \overline{OC}^2 h$
9. $= \frac{2}{3}\pi (R^2 - \overline{OC}^2) h$.
10. But, $R^2 - \overline{OC}^2 = \overline{AC}^2$ Why?
11. $= (\frac{1}{2} AB)^2$
12. $= \frac{1}{4}\overline{AB}^2$.
13. $\therefore$ vol. $ACDB = \frac{2}{3}\pi \times \frac{1}{4}\overline{AB}^2 \times h = \frac{1}{6}\pi \overline{AB}^2 h$.
14. Now, $\overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2$
15. $= (r - r')^2 + h^2$.
16. $\therefore$ vol. $ACDB = \frac{1}{6}\pi[(r - r')^2 + h^2]h$.
17. Also, vol. $ABB'A' = \frac{1}{3}\pi(r^2 + r'^2 + rr')h$. § 600
18. Substituting in step 2, vol. $ADBB'A'$
    $= \frac{1}{6}\pi[(r - r')^2 + h^2]h + \frac{1}{6}\pi(2r^2 + 2r'^2 + 2rr')h$
19. $= \frac{1}{6}\pi(r^2 - 2rr' + r'^2 + h^2 + 2r^2 + 2r'^2 + 2rr')h$
20. $= \frac{1}{6}\pi h\{3r^2 + 3r'^2 + h^2\}$.

Ex. 98. Find the volume of a spherical segment, the radii of whose bases are 4 and 5, and whose altitude is 9.

Ex. 99. A hemisphere is a segment of one base whose altitude $h = r$. Find the volume of the hemisphere by substituting r for h and letting $r' = 0$.

Ex. 100. In a sphere of radius 10 in., a plane is passed parallel to the "equator," and 5 in. from the center. Find the volume of the spherical segment between this plane and the plane of the equator.

691. A **spherical wedge** is a solid bounded by a lune and the planes of its bounding arcs.

Evidently, two spherical wedges in the same sphere, or equal spheres, are congruent when their angles are equal.

Also, it is evident that two wedges in the same sphere can be added by placing them so that they have one common bounding plane. The angle of the sum is equal to the sum of the angles of the wedges.

692. It can be proved as in § 658 that two wedges have the same ratio as their angles.

693. A sphere may be regarded as a wedge whose angle is 360°. (Cf. § 659.)

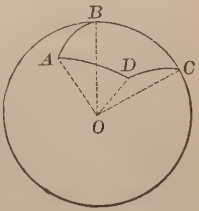
In a sphere whose radius is r, a wedge whose angle contains A degrees has a volume v determined as follows:

$$\frac{v}{\frac{4}{3}\pi r^3} = \frac{A}{360}, \text{ or } v = \frac{\pi r^3 A}{270}. \qquad \text{§ 675}$$

694. The area of the lune whose angle is A degrees, on a sphere whose radius is r, is $\frac{\pi r^2 A}{90}$. § 661

∴ *the volume of a wedge equals one third the radius of the sphere multiplied by the area of the lune which forms its base.*

695. A **spherical pyramid** is a solid bounded by the surface inclosed by a spherical polygon and the planes of its sides; as O–$ABCD$ in the adjoining figure.

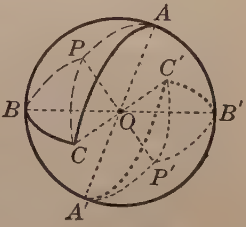


The center of the sphere is the *vertex* of the pyramid, and the spherical polygon is its *base.*

Two *spherical pyramids are congruent* when their bases are congruent, for they can be made to coincide.

696. *Two spherical pyramids whose bases are symmetrical isosceles spherical triangles are congruent, for their bases are congruent by* § 663.

697. *Two spherical pyramids corresponding to a pair of vertical triedral angles are equal.* (Cf. § 666.)



Suggestions. 1. Recall the proof of § 665.

2. Compare spherical pyramids O–APB, O–BPC, and O–CPA with spherical pyramids O–$A'P'B'$, O–$B'P'C'$, and O–$C'P'A'$, respectively, using § 696.

698. *The volume of a triangular spherical pyramid equals one half the volume of a spherical wedge whose angle is the spherical excess of the base of the pyramid.*

The proof is exactly like that for § 668.

699. *If the radius of the sphere is r and the excess of the base of a triangular spherical pyramid is E, and the volume of the spherical pyramid is v, then*

$$v = \frac{1}{2}\,\frac{\pi r^3 \cdot E}{270} = \frac{\pi r^3 E}{540}. \qquad \text{§ 693}$$

700. The same formula may be employed to find the volume of any spherical pyramid, with the understanding that E is the spherical excess of the base of the pyramid, measured in degrees.

701. In the case of any spherical pyramid, the area of the base is $\frac{\pi r^2 E}{180}$ (§ 670). Hence the volume of any spherical pyramid is one third the area of its base multiplied by the radius of the sphere.

Ex. 101. Find the volume of a triangular spherical pyramid the angles of whose base are 92°, 119°, and 134°, if the volume of the sphere is 192.

Ex. 102. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107°, 118°, 134°, and 146°, if the diameter of the sphere is 12.

Ex. 103. The volume of a triangular spherical pyramid, the angles of whose base are 105°, 126°, and 147°, is $60\frac{1}{2}$. What is the volume of the sphere?

Ex. 104. Find the volume of a pentagonal spherical pyramid the angles of whose base are 109°, 128°, 137°, 153°, and 158°, on the sphere of radius 12 in.

Ex. 105. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 110°, 122°, 135°, and 146° on the sphere whose radius is 10 in.

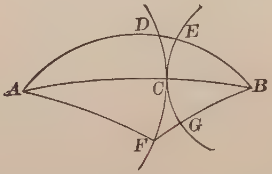
Ex. 106. What is the angle of the base of a spherical wedge whose volume is $\frac{40}{3}\pi$, if the radius of the sphere is 4?

Ex. 107. What is the volume of the spherical pyramid whose base is a trirectangular triangle in a sphere of radius r?

OPTIONAL TOPIC D

PROPOSITION XXIX. THEOREM

702. *The shortest line on the surface of a sphere between two given points is the arc of the great circle, not greater than a semicircle which joins the two points.*



Hypothesis. Points A and B are on the surface of a sphere, and AB is an arc of a great ⊙, not greater than a semicircle.

Conclusion. AB is the shortest line on the surface of the sphere between A and B.

Proof: 1. Let C be any point on AB. Through C, draw small circles DCF and ECG, having A and B, respectively as poles.

2. Let F be any point on $\odot DCF$ except C, and AF and BF be arcs of great circles.

$$\text{Then } \overset{\frown}{AF} = \overset{\frown}{AC}, \text{ and } \overset{\frown}{AF} + \overset{\frown}{BF} > \overset{\frown}{AC} + \overset{\frown}{CB}.$$

$$\therefore \overset{\frown}{BF} > \overset{\frown}{BC}, \text{ or } \overset{\frown}{BF} > \overset{\frown}{BG}, \text{ and } F \text{ lies outside } \odot ECG.$$

$$\therefore \text{ every point of } \overset{\frown}{DCF} \text{ except } C \text{ lies outside } \odot ECG.$$

3. Let $ADEB$ be any curve joining A and B, but not through C, cutting $\odot DCF$ at D and $\odot ECG$ at E.

4. Let curve AD be revolved about A until D coincides with C; then there is a line from A to C equal to AD. Similarly a line can be secured from B to C equal to BE. Hence there is a line from A to B, through C, which equals $AD + EB$. But this line is less than $ADEB$ by the amount DE.

5. $\therefore$ any line from A to B, not through C, is longer than a line from A to B through C.

6. But C was any point on AB. Hence any line from A to B other than AB is longer than AB.

Ex. 1. If a plane contains a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

Hyp. $ABCD$ is a ▱.
BG and $DH \perp$ plane $AECF$.
Con. $BG = DH$.

Ex. 2. If three or more planes intersect in pairs, their lines of intersection are either parallel or concurrent.

Plan. *Case 1.* Assume that two lines of intersection meet at P, and prove that P must lie in the third line of intersection.
Case 2. Assume that two lines of intersection are parallel, and prove that the third line of intersection must be parallel to them by the indirect method of proof.

Ex. 3. Prove that there is, in general, one straight line through a given point which will intersect each of two given skew straight lines.

Plan. Let the second line cut the plane determined by the first line and the point. Draw the line from the foot of the second line through the given point.

Ex. 4. Prove that, in general, a straight line can be drawn which will intersect each of three given straight lines, each of which is skew to each of the others.

Plan. Pass a plane through the first line, cutting the other two lines. Draw a line through the feet of the second and third lines.

Ex. 5. Prove that a line can be drawn perpendicular to each of two parallel lines, and intersecting a third line which is not parallel to them.

Ex. 6. Prove that, in general, a straight line can be drawn so as to intersect each of two given straight lines and be parallel to the third, if the lines are skew each to each.

Plan. Pass a plane through one line parallel to the third, and cutting the second. Through the foot of the second, draw a line parallel to the third.

Ex. 7. If a plane, parallel to the edge of a diedral angle, intersects the faces of the angle, its intersections with the faces are parallel to the edge of the angle and to each other.

Ex. 8. Prove that all lines which intersect a given straight line and are also parallel to a second straight line lie in a plane containing the first line. (See § 431.)

Ex. 9. Prove that a line parallel to each of two intersecting planes is parallel to their intersection.

Plan. Pass a plane through the line and parallel to one of the given planes.

Ex. 10. If each of two intersecting planes is cut by two parallel planes which are themselves not parallel to the intersection of the given planes, the angles formed in the parallel planes are equal.

Ex. 11. (*a*) Prove that a straight line can be drawn intersecting each of two skew straight lines and parallel to a given plane, which does not contain either of the lines.

(*b*) How many such lines can be drawn?

Plan. Pass a plane parallel to the given plane, cutting the two given straight lines.

Ex. 12. D is any point in perpendicular AF from A to side BC of $\triangle ABC$. If line DE be drawn perpendicular to the plane ABC, and line GH be drawn through E parallel to BC, then line AE is perpendicular to GH.

Suggestion. Prove $BC \perp$ plane AED, and then prove $GH \perp AED$.

Ex. 13. In one face of a diedral angle a line AV is drawn intersecting the edge at V. Prove that a line BV can be drawn in the other face so that the $\angle AVB$ is a right angle.

Plan. Through V, pass a plane $\perp AV$.

Ex. 14. Prove that a plane can be passed through the four faces of a tetraedral angle so that the intersections with the faces is a parallelogram.

Plan. One pair of opposite faces intersects in a line through the vertex; the other pair does also, giving two lines through the vertex. Pass a plane, cutting the faces of the tetraedral angle, parallel to each of these lines.

Ex. 15. The sum of the perpendiculars to a plane from the vertices of a triangle which lies outside the plane equals three times the perpendicular to the plane from the center of gravity of the triangle.

Ex. 16. If a line segment is not parallel to a plane, it is longer than its projection on the plane.

Ex. 17. If the projection of AB on a plane is 6 in. long, and AB is 12 in. long, what is the inclination of AB to the plane?

Ex. 18. What is the length of the projection on a plane of a segment 42 in. long if the inclination of it to the plane is 60°? 30°? 45°?

Ex. 19. If two segments drawn to a plane from an outside point make equal angles with the plane, they are equal.

Ex. 20. Given a circle which lies in a plane parallel to a given plane. Prove that its projection on the plane is a circle equal to the given circle.

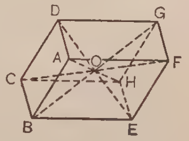
Ex. 21. An equilateral triangle $\triangle ABC$, whose area is 75 sq. in., has its base AB parallel to a plane PQ. The plane of ABC makes an angle of 45° with the plane PQ. Find the area of the projection of $\triangle ABC$ on plane PQ.

Ex. 22. The base of rectangle $ABCD$ is 20 in. and its altitude is 12 in. The side of length 20 in. is parallel to a plane QR. The side of length 12 in. makes an angle of 60° with the plane QR. Find the area of the projection of $ABCD$ on the plane QR.

Ex. 23. If the opposite face angles of a tetraedral angle are equal, then the opposite diedral angles also are equal.

Ex. 24. Prove that the sum of the squares of the four diagonals of a parallelepiped is equal to the sum of the squares of its twelve edges.

Suggestion. Use § 311, and § 312 in ▱s $DGEB$, $ADHE$, and $GFBC$.



Ex. 25. Find the lateral area of a right triangular prism each side of whose base is 5 and whose altitude is 8.

Ex. 26. Determine the approximate area of the base of a bin 6 ft. deep that will hold 250 bu. of grain. (One bu. = 2150.42 cu. in.)

Ex. 27. Find the edge of a cube equal to a rectangular parallelepiped whose dimensions are 9 in., 1 ft. 9 in., and 4 ft. 1 in.

Ex. 28. Find the volume of a rectangular parallelepiped, the dimensions of whose base are 14 and 9, and the area of whose entire surface is 620.

Ex. 29. The diagonal of a cube is $8\sqrt{3}$. Find its volume, and the area of its entire surface.

Suggestion. Represent the length of the edge by x.

Ex. 30. Find the area of the entire surface of a rectangular parallelepiped, the dimensions of whose base are 11 and 13, and volume 858.

Ex. 31. A trench is 124 ft. long, $2\frac{1}{2}$ ft. deep, 6 ft. wide at the top, and 5 ft. wide at the bottom. How many cubic feet of water will it contain?

Ex. 32. Prove that the volume of any oblique prism is equal to the product of the area of a right section by the length of a lateral edge.

Ex. 33. Prove that the volume of a right prism, having its base inclosed by a regular polygon, equals its lateral area multiplied by one half the apothem of the base.

Ex. 34. The volume of a right prism is 2310, and its base is inclosed by a right triangle whose legs are 20 and 21, respectively. Find its lateral area.

Ex. 35. What is the volume of the right prism, the sides of whose base are 13, 14, and 15, and whose altitude is 20?

Ex. 36. A plane is passed parallel to and 5 in. from the base of a square pyramid whose base edge is 4 in. and whose altitude is 20 in. What is the area of the section parallel to the base?

Ex. 37. The altitude of a pyramid is 20 in., and its base is inclosed by a rectangle whose dimensions are 10 in. and 15 in., respectively. What is the distance from the vertex of a section parallel to the base, whose area is 54 sq. in.?

Ex. 38. At what distance from the vertex must a plane parallel to the base be drawn so that the area of the section will be one half the base?

Ex. 39. In Ex. 38 replace the fraction $\frac{1}{2}$ by the fraction $\frac{1}{3}$ and solve the resulting exercise.

Ex. 40. In Ex. 38 replace the fraction $\frac{1}{2}$ by the fraction $\frac{2}{3}$ and solve the resulting exercise.

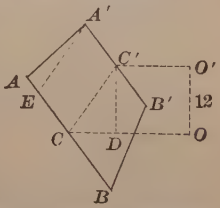
Ex. 41. Prove that the volume of a regular pyramid is equal to its lateral area, multiplied by one third the distance from the center of its base to any lateral face.

Suggestion. 1. Pass planes through the lateral edges and the center of the base.

2. Prove that the center of the base is equidistant from the lateral faces.

Ex. 42. Find the lateral edge, lateral area, and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 20 and 10, respectively, and whose altitude is 12.

Suggestion. Let $ABB'A'$ be a lateral face of the frustum, and O and O' the centers of the bases; draw $OC \perp AB$, $O'C' \perp A'B'$, $C'D \perp OC$, and $A'E \perp AB$; also lines OO' and CC'.



Ex. 43. A pyramid whose base is inclosed by a regular hexagon of side 5 in. and whose altitude is 25 in. is cut by a plane, parallel to its base and 5 in. from the base. What is the volume of the frustum formed?

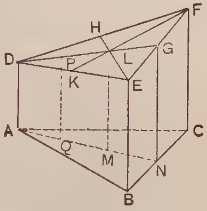
Ex. 44. Find the volume of the rectangular prismatoid the sides of whose bases are 10 and 7, and 6 and 5, respectively, and whose altitude is 9.

Ex. 45. The volume of a triangular prism is equal to a lateral face, multiplied by one half its perpendicular distance from any point in the opposite lateral edge.

Suggestion. Draw a rt. section of the prism, and apply § 543.

Ex. 46. Prove that the perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base, is equal to one third the sum of the lateral edges.

Suggestion. Let P be the mid-point of DL, and draw $PQ \perp ABC$; express LM in terms of PQ and GN.

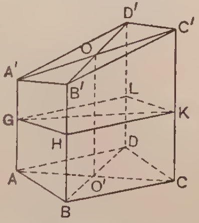


Ex. 47. Prove that the sum of two opposite lateral edges of a truncated parallelepiped is equal to the sum of the other two lateral edges.

Suggestions. 1. What kind of figure is $AA'C'C$?

2. Compare $AA' + CC'$ with OO'.

Ex. 48. Prove that the upper base of a truncated parallelepiped is inclosed by a parallelogram.



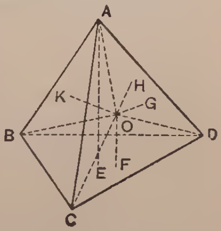
Ex. 49. Prove that a plane passed through the center of a parallelepiped divides it into two equal solids.

Ex. 50. The volume of a truncated parallelepiped is equal to the area of a right section, multiplied by the distance between the centers of the bases.

Suggestion. By Ex. 47, the distance between the centers of the bases may be proved equal to one fourth the sum of the lateral edges.

Ex. 51. The sum of the perpendiculars drawn to the faces from any point within a regular tetraedron is equal to its altitude.

Suggestion. Divide the tetraedron into triangular pyramids, having the given point for their common vertex. Find the volume of each and of the whole pyramid and form an equation.



Ex. 52. Prove that the volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

Ex. 53. What is the locus of points in space at the distance d from an unlimited line l?

Ex. 54. The lateral area of a cylinder of revolution is 120π. The area of the base is 36π. Find the altitude.

Ex. 55. Determine the lateral area of the right circular cylinder formed by revolving a rectangle, having base b and altitude h:

(*a*) about its base; (*b*) about its altitude.

Ex. 56. How many square feet of heating surface are there in a hot-water pipe 9 feet long and 2 inches in outside diameter?

Ex. 57. How high must a cylindrical can 5 in. in diameter be in order that it contain one gallon (231 cu. in.)?

Ex. 58. What is the volume of the right circular cylinder inscribed in a cube of edge 10 in.?

Ex. 59. What is the volume of the right circular cylinder circumscribed about a right hexagonal prism of altitude 12 in., whose base is a regular hexagon having sides 2 in. long?

Ex. 60. What is the effect upon the volume of a right circular cylinder:

(*a*) if the altitude is doubled and the radius remains unchanged?

(*b*) if the altitude remains unchanged and the radius is doubled? trebled?

Ex. 61. Find the lateral area, the total area, and the volume of the right circular cylinder whose:

(*a*) altitude is 8 in., and radius is 4 in.

(*b*) altitude is 12 in., and diameter is 6 in.

Ex. 62. Prove that the volume of a cylinder of revolution is equal to its lateral area multiplied by one half the radius of its base.

Ex. 63. A regular hexagonal prism is inscribed in a right circular cylinder whose altitude is 10 in. and the radius of whose base is 3 in. Determine the difference between the volumes of the prism and cylinder.

Ex. 64. Two right circular cylinders have equal altitudes, but the radius of the base of the one is double the radius of the base of the other. Compare (*a*) their lateral areas; (*b*) their volumes.

Ex. 65. What must be the length in inches of a 10-gal. gasoline tank which is 10 in. in diameter?

Ex. 66. The cross section of a tunnel, $2\frac{1}{2}$ mi. in length, is in the form of a rectangle 6 yd. wide and 4 yd. high, surmounted by a semicircle whose diameter is equal to the width of the rectangle; how many cubic yards of material were taken out in its construction? ($\pi = 3.1416$.)

Ex. 67. Express by a formula the volume of a round cast-iron column of length l ft., thickness t in., and outside diameter d in.

Ex. 68. (*a*) A right triangle whose legs are c and d respectively is revolved about the leg c. What is the volume of the cone formed?

(*b*) It is next revolved about the leg d. What is the volume of the cone formed?

(*c*) What is the ratio of the volumes?

Ex. 69. A right triangle whose hypotenuse is 16 in. has one acute angle of 60°. It is revolved around the side which lies opposite this 60° angle.

(*a*) What is the volume of the cone generated?

(*b*) What is the lateral area of this same cone?

Ex. 70. An isosceles right triangle whose sides are a in. long is revolved about one of these sides as axis. What is the volume and the lateral area of the cone generated?

Ex. 71. If the altitude of a cone remains unchanged, but its radius be trebled, what is the effect on the:

(*a*) slant height? (*b*) volume? (*c*) lateral area?

Ex. 72. The slant height of a right circular cone is 12 in. and the diameter of its base is 6 in. What is its:

(*a*) lateral area? (*b*) total area? (*c*) volume?

Ex. 73. What is the volume of the regular hexagonal pyramid inscribed in the cone of Ex. 72?

Ex. 74. The altitude of a cone of revolution is 27 in., and the radius of its base is 16 in. What is the diameter of the base of an equal cylinder, whose altitude is 16 in.?

Ex. 75. What is the effect upon the volume of a circular cone:

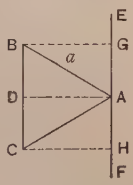
(*a*) if the radius of the base is doubled, and the altitude is constant?

(*b*) if the altitude is doubled, and the radius is constant?

(*c*) if both the radius and the altitude are doubled?

Ex. 76. A portion of a plane bounded by an equilateral triangle, whose side is a, revolves about a straight line drawn through one of its vertices parallel to the opposite side. Find the area of the entire surface, and the volume of the solid generated.

(The solid generated is the difference of the cylinder generated by $BCHG$, and the cones generated by ABG and ACH.)



Ex. 77. A portion of a plane bounded by an equilateral triangle, whose altitude is h, revolves about one of its altitudes as an axis. Find the area of the surface, and the volume of the solid generated.

Ex. 78. A portion of a plane bounded by a right triangle, whose legs are 4 and 10, revolves about its hypotenuse as an axis. Find the area of the entire surface, and the volume of the solid generated.

Ex. 79. A portion of a plane bounded by an equilateral triangle, whose side is 6, revolves about one of its sides as an axis. Find the area of the entire surface, and the volume of the solid generated.

Ex. 80. What is the volume of the frustum of the right circular cone, whose radii are 3 in. and 6 in., respectively, and:

(*a*) whose altitude is 10 in.?

(*b*) whose slant height is 10 in.?

Ex. 81. (*a*) What is the difference in volumes of a right circular cone of altitude h and radius r, and the regular hexagonal pyramid inscribed in it?

(*b*) What is the value of this result when $h = 15$ and $r = 4$?

Ex. 82. What is the ratio of:

(*a*) the lateral areas of two similar cylinders of revolution if their altitudes are as 2 : 5?

(*b*) What is the ratio of the volumes of these same cylinders?

Ex. 83. Using Note, p. 412, if an army of given size is represented by the figure of a soldier 1 in. high, how large an army is represented by a similar figure 2 in. high?

Ex. 84. If the number of bushels of corn produced in one year is represented by an ear of corn of length l, how long should a similar ear of corn be made to represent 9 times as many bushels?

Ex. 85. What is the locus of the center of a sphere which will pass through all the points of a given circle?

Ex. 86. What is the locus of the center of a sphere which will pass through each of three given points?

Ex. 87. What is the locus of the center of a sphere which will pass through each of two given points?

Ex. 88. How far from the center of a sphere of radius 10 in. is a small circle of it whose circumference is 8π in.?

Ex. 89. What is the radius of the circle made by a plane 3 in. from the center of a sphere of radius 10 in.?

Ex. 90. The polar distance of a circle of a sphere of radius 12 in. is 30°. What is the radius of the circle and what is its distance from the center of the sphere?

Ex. 91. Repeat Ex. 90, if the angle is 60°.

Ex. 92. Repeat Ex. 90, if the angle is 45°.

Ex. 93. Is it possible for a sphere to be tangent to each of the faces of a given triedral angle? If there is more than one such sphere, what is the locus of the center?

Ex. 94. What is the locus of the center of a sphere which will be tangent to each of the faces of a given diedral angle?

Ex. 95. What is the locus of the center of a sphere which will have a given radius r and will be tangent to a given plane?

Ex. 96. What is the locus of the center of a sphere which is tangent to a given plane at a given point?

Ex. 97. Prove that a sphere can be circumscribed about a cube.

Ex. 98. Is it possible for a sphere to pass through all the points of a given circle and also through a given point outside the plane of the circle? If so, tell how to determine its center and its radius.

Ex. 99. What is the spherical excess of $\triangle ABC$, if:

(*a*) $\angle A = 75^\circ$; $\angle B = 1\frac{1}{2}$ rt. $\angle$; $\angle C = 150^\circ$?
(*b*) $\angle A = \frac{2}{3}$ rt. $\angle$; $\angle B = \frac{3}{5}$ st. $\angle$; $\angle C = 60^\circ$?
(*c*) $\angle A = 85^\circ$; $\angle B = 95^\circ$; $\angle C = 105^\circ$?

Ex. 100. What is the spherical excess of the quadrilateral $ABCD$, if:

(*a*) $\angle A = 60^\circ$; $\angle B = 70^\circ$; $\angle C = 125^\circ$; $\angle D = 140^\circ$?
(*b*) $\angle A = 85^\circ$; $\angle B = 90^\circ$; $\angle C = 100^\circ$; $\angle D = 125^\circ$?

Ex. 101. What is the polar triangle of a trirectangular triangle?

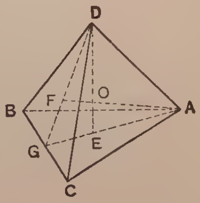
Ex. 102. What kind of a triangle is the polar triangle of a birectangular triangle?

Ex. 103. (*a*) What is the area of the zone of altitude 4 in. on a sphere of radius 10 in.?

(*b*) Repeat the exercise if $h = 5$ in.
(*c*) Repeat the exercise if $h = 8$ in.

Ex. 104. Find the area of the spherical surface passing through the vertices of a regular tetraedron whose edge is 8.

Suggestion. Draw DOE amd AOF perpendicular to $\triangle$s ABC and BCD respectively.



Ex. 105. What part of the surface of the sphere is the lune whose angle is: (*a*) 30°? (*b*) 72°? (*c*) 100°?

Ex. 106. How large is the angle of the lune which equals a trirectangular triangle of the sphere?

Ex. 107. How large is the angle of the lune which equals the triangle whose angles are 70°, 80° and 90°?

Ex. 108. On the sphere of area 100 sq. in., what is the area of the lune whose angle is: (*a*) 50°? (*b*) 85°? (*c*) 125°?

Ex. 109. On a sphere whose area is 260 sq. in., what is the area of the spherical triangle whose excess E is: (*a*) 50°? (*b*) 75°? (*c*) 100°?

Ex. 110. What part of the sphere is inclosed by the triangle whose angles are 75°, 100°, and 125°?

Ex. 111. What is the volume of the sphere whose radius is: (*a*) 5 in.? (*b*) 2 ft.? (*c*) 13 in.?

Ex. 112. What is the radius of the sphere whose volume is 1 cu. foot, expressed correct to the nearest tenth of an inch?

Ex. 113. A circular sector whose central angle is 45° and radius 12 revolves about a diameter perpendicular to one of its bounding radii. Find the volume of the spherical sector generated.

Ex. 114. Given the volume of a sphere V to find the area of its surface.

Ex. 115. What is the ratio of the volumes and of the areas of two spheres whose radii are 5 in. and 8 in. respectively?

Ex. 116. What is the ratio of the volumes of two spheres, if the ratio of their areas is 36 : 81?

Ex. 117. Find the volume of a spherical sector, the altitude of whose base is 12, the diameter of the sphere being 25.

Ex. 118. If opposite sides of a spherical quadrilateral are equal, the opposite angles also are equal.

Ex. 119. Prove that the diagonals of the quadrilateral in Ex. 118 bisect each other.

AXIOMS, POSTULATES, DEFINITIONS, THEOREMS FROM PLANE GEOMETRY

§ **5.** (*a*) One and only one straight line can be drawn through two points; or two points determine a straight line.

(*b*) A straight line extends infinitely far in each of two directions.

§ **9.** Two straight lines can intersect at only one point.

§ **13.** The straight line-segment is the shortest line between two points.

§ **16.** A circle is a closed curved line, lying in a plane, such that all points of it are equidistant from a point within called the center.

§ **19.** All radii of the same circle or equal circles are equal.

§ **21.** An angle is the figure formed by two rays drawn from the same point.

§ **24.** Adjacent angles are two angles that have a common vertex and a common side between them.

§ **26.** If one straight line meets another straight line so that the adjacent angles formed are equal, each of these angles is a right angle.

§ **27.** All right angles are equal.

§ **33.** Two lines are perpendicular if they form a right angle.

§ **35.** The sum of all the successive adjacent angles around a point on one side of a straight line through the point is one straight angle, or 180°.

§ **36.** The sum of all the successive adjacent angles around a point in a plane is two straight angles, or 360 degrees.

§ **37.** Two angles are complementary if their sum is equal to a right angle.

§ **38.** Complements of the same angle or of equal angles are equal.

§ **39.** Two angles are supplementary if their sum is equal to a straight angle.

§ **40.** If two adjacent angles have their exterior sides in a straight line, they are supplementary.

§ **41.** If two adjacent angles are supplementary, their exterior sides lie in a straight line.

§ **42.** Supplements of the same angle or of equal angles are equal.

§ **49.** **Ax. 1.** Quantities which are equal to the same quantity or to equal quantities are equal to each other.

Ax. 2. Any quantity may be substituted for its equal in an equation or an inequality.

Ax. 3. If equals be added to equals, the sums are equal.

Ax. 4. If equals be subtracted from equals, the remainders are equal.

Ax. 5. If equals be multiplied by equals, the products are equal.

Ax. 6. If equals be divided by equals, the quotients are equal.

Ax. 7. The whole equals the sum of its parts.

Ax. 8. The whole is greater than any of its parts.

§ **59.** Geometrical figures are congruent if they can be made to coincide.

§ **60.** *Postulate.* A geometrical figure may be moved without changing its shape or the size of its parts.

§ **62.** If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.

§ **66.** If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.

§ **68.** The base angles of an isosceles triangle are equal.

§ **70.** An equilateral triangle is also equiangular.

§ **71.** The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base.

§ **73.** If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.

§ **77.** If two points are each equidistant from the ends of a segment, they determine the perpendicular-bisector of the segment.

§ **80–81.** One and only one perpendicular can be drawn to a line at a point in the line.

§ **86.** An exterior angle of a triangle is greater than either remote interior angle.

§ **87.** There can be only one perpendicular to a given line from a point not on the line.

§ **88.** Two straight lines are parallel if they lie in the same plane and do not meet even if extended.

§ **89.** *Postulate of Parallels.* Through a given point there can be only one parallel to a given line.

§ **90.** If two lines are parallel to a third line, they are parallel to each other.

§ **92.** If two lines are cut by a transversal so that a pair of alternate-interior angles are equal, the lines are parallel.

§ **95.** If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.

§ **96.** If two lines are perpendicular to a third line, they are parallel.

§ **97.** If two lines are cut by a transversal so that a pair of interior angles on the same side of the transversal are supplementary, the lines are parallel.

§ **99.** If two parallels are cut by a transversal, alternate-interior angles are equal.

§ **100.** If two parallels are cut by a transversal, corresponding angles are equal.

§ **101.** If a line is perpendicular to one of two parallels, it is perpendicular to the other also.

§ **102.** If two parallels are cut by a transversal, interior angles on the same side of the transversal are supplementary.

§ **103.** If two angles have their sides respectively parallel, they are equal, provided both pairs of parallels extend in the same directions from the vertices, or in opposite directions.

§ **105.** The sum of the angles of any triangle is one straight angle.

§ **107.** A triangle cannot have two right angles or two obtuse angles.

§ **108.** The acute angles of a right triangle are complementary.

§ **109.** An exterior angle of a triangle equals the sum of the two remote interior angles.

§ **110.** If two angles of one triangle equal respectively two angles of another triangle, the third angles are equal.

§ **111.** If two triangles have a side, the opposite angle, and another angle of the one equal respectively to a side, the opposite angle, and another angle of the other, the triangles are congruent.

§ **112.** If two angles have their sides respectively perpendicular, they are either equal or supplementary.

§ **113.** If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent.

§ **116.** **I.** Any point in the perpendicular-bisector of a segment is equidistant from the ends of the segment.

II. Any point equidistant from the ends of a segment lies in the perpendicular-bisector of the segment.

§ **117.** **I.** Any point in the bisector of an angle is equidistant from the sides of the angle.

II. Any point equidistant from the sides of an angle lies in the bisector of the angle.

§ **118.** If two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.

§ **119.** If a triangle is equiangular, it is also equilateral.

§ **120.** A polygon is a closed broken line in a plane.

§ **121.** A polygon is convex if no side, when extended, will pass through the interior of the polygon.

§ **125.** A diagonal of a parallelogram divides it into two congruent triangles.

§ **126.** The opposite sides of a parallelogram are equal.

§ **127.** The opposite angles of a parallelogram are equal.

§ **128.** Two consecutive angles of a parallelogram are supplementary.

§ **129.** Segments of parallels included between parallels are equal.

§ **130.** Two parallels are everywhere equidistant.

§ **131.** The diagonals of a parallelogram bisect each other.

§ **133.** If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

§ **134.** If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

§ **135.** If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

§ **141.** If three or more parallels intercept equal segments on one transversal, they intercept equal segments on any other transversal.

§ **142.** If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side also.

§ **143.** If a line is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, it bisects the other also.

§ **145.** The sum of the interior angles of a polygon having n sides is $(n - 2)$ straight angles.

§ **148.** Axioms for combining inequalities.

Ax. 9. If equals be added to unequals, the sums are unequal in the same order.

Ax. 10. If equals be subtracted from unequals, the differences are unequal in the same order.

Ax. 11. If $a > b$ and $b > c$, then $a > c$.

Ax. 12. If unequals be added to unequals in the same order, the sums are unequal in the same order.

Ax. 13. If unequals be subtracted from equals or from unequals of opposite order, the differences are unequal and of order opposite to that of the subtrahend.

§ **149.** Fundamental inequalities for segments.

(*a*) Any side of a triangle is less than the sum of the other two sides.

(*b*) Any side of a triangle is greater than the difference of the other two sides.

§ **150.** Fundamental inequality for angles.

An exterior angle of a triangle is greater than either remote interior angle of the triangle.

§ **151.** If two sides of a triangle are unequal, the angles opposite are unequal, the angle opposite the greater side being the greater.

§ **152.** If two angles of a triangle are unequal, the sides opposite are unequal, the side opposite the greater angle being the greater.

§ **153.** The hypotenuse of a right triangle is greater than either leg of the triangle.

§ **154.** The perpendicular from a point to a line is the shortest segment from the point to the line.

§ **155.** If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

§ **156.** If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

§ **159.** If a segment joins the mid-points of two sides of a triangle, it is parallel to the third side and equal to one half of it.

§ **161.** The median of a trapezoid is parallel to the bases and equal to one half their sum.

§ **163.** The bisectors of the interior angles of a triangle meet at a point which is equidistant from the sides of the triangle.

§ **164.** The perpendicular-bisectors of the sides of a triangle meet at a point which is equidistant from the vertices of the triangle.

§ **165.** The altitudes of a triangle meet at a point.

§ **166.** The medians of a triangle meet at a point which lies two thirds the distance from each vertex to the mid-point of the opposite side.

§ **172.** The following facts are now evident:

(*a*) All radii and all diameters of the same circle or of equal circles are equal.

(*b*) A point is within, on, or outside a circle according as its distance from the center is less than, equal to, or greater than the radius.

(*c*) A diameter of a circle bisects the circle and the surface inclosed by it; also, if a line bisects a circle, it is a diameter.

§ **174.** Only one circle can be drawn through three points which are not in a straight line.

§ **175.** A circle cannot be drawn through three points which do lie in a straight line.

§ **176.** A straight line cannot intersect a circle in more than two points.

§ **180.** In the same circle or in equal circles, if central angles are equal, they intercept equal arcs.

§ **181.** In the same circle or in equal circles, if arcs are equal, the central angles which intercept them are equal.

§ **182.** It may be proved by superposition that: in the same circle or in equal circles:

(*a*) The greater of two unequal central angles intercepts the greater arc;

(*b*) The greater of two unequal arcs is intercepted by the greater central angle.

§ **184.** In the same circle or in equal circles, if chords are equal, they subtend equal arcs.

§ **185.** In the same circle or in equal circles, if arcs are equal, the chords which subtend them are equal.

§ **187.** In the same circle or in equal circles, if two minor arcs are unequal, then their chords are unequal, the greater arc being subtended by the greater chord.

§ **188.** In the same circle or in equal circles, if two chords are unequal, then they subtend unequal minor arcs, the greater chord subtending the greater arc.

§ **189.** If a diameter is perpendicular to a chord, it bisects the chord and its subtended arcs.

§ **190.** The perpendicular-bisector of a chord passes through the center of the circle.

§ **191.** In the same circle or in equal circles, if chords are equal, they are equidistant from the center.

§ **192.** In the same circle or in equal circles, if chords are equidistant from the center, they are equal.

§ **194.** In the same circle or in equal circles, the less of two unequal chords is at the greater distance from the center of the circle.

§ **195.** In the same circle or in equal circles, if two chords are unequally distant from the center, the more remote is the smaller.

§ **197.** A straight line perpendicular to a radius at its outer extremity is tangent to the circle.

§ **198.** A tangent to a circle is perpendicular to the radius drawn to the point of contact.

§ **199.** A line perpendicular to a tangent at its point of contact passes through the center of the circle.

§ **200.** A line from the center of the circle perpendicular to a tangent passes through the point of contact.

§ **201.** The tangents to a circle from an outside point are equal.

§ **202.** The two tangents to a circle from an outside point make equal angles with the line from that point to the center of the circle.

§ **203.** Parallel lines intercept equal arcs on a circle.

§ **205.** Two magnitudes of the same kind are commensurable when each contains the same unit of measure, called a common measure, an integral number of times.

Two magnitudes of the same kind are incommensurable when no unit of measure can be found which is contained an integral number of times in each.

§ **207.** In the same circle or in equal circles, two central angles have the same ratio as their intercepted arcs.

§ **210.** A central angle is measured by its intercepted arc.

§ **212.** An inscribed angle is measured by one half its intercepted arc.

§ **213.** An angle inscribed in a semicircle is a right angle.

§ **214.** Inscribed angles which intercept the same arc are equal.

§ **215.** The angle formed by a tangent and a chord drawn to the point of contact is measured by one half its intercepted arc.

§ **216.** The angle formed by two chords intersecting within a circle is measured by one half the sum of the arcs intercepted by it and its vertical angle.

§ **217.** The angle formed by two secants intersecting outside the circle is measured by one half the difference between its intercepted arcs.

§ **218.** The angle formed by a secant and a tangent is measured by one half the difference between its intercepted arcs.

§ **219.** The angle formed by two tangents is measured by one half the difference between its intercepted arcs.

§ **233.** If two circles are tangent to each other, their line of centers passes through their point of contact.

§ **234.** If the distance between the centers of two circles equals the sum of their radii, the circles are tangent externally.

§ **235.** If two circles intersect, the straight line joining their centers bisects their common chord at right angles.

§ **250.** In a proportion, the product of the extremes is equal to the product of the means.

§ **251.** The mean proportional between two numbers is the square root of their product.

§ **252.** If the product of two numbers is equal to the product of two other numbers, either pair may be made the means of a proportion of which the other pair are the extremes.

§ **253.** If the antecedents of a proportion are equal, then also the consequents are equal; if the consequents are equal, then also the antecedents are equal.

§ **254.** If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, the remaining terms also are equal.

§ **255.** In any proportion, the terms are in proportion by alternation; that is, the first is to the third as the second is to the fourth.

§ **256.** In any proportion, the terms are in proportion by inversion: that is, the second is to the first as the fourth is to the third.

§ **257.** In any proportion, the terms are in proportion by addition: that is, the first plus the second is to the second as the third plus the fourth is to the fourth.

§ **258.** In any proportion, the terms are in proportion by subtraction: that is, the first minus the second is to the second as the third minus the fourth is to the fourth.

§ **259.** In any proportion, the terms are in proportion by addition and subtraction; that is, the first plus the second is to the first minus the second as the third plus the fourth is to the third minus the fourth.

§ **262.** A parallel to one side of a triangle, intersecting the other two sides, divides the other two sides proportionally.

§ **267.** Parallel lines intercept proportional segments on all transversals.

§ **269.** A line which divides two sides of a triangle proportionally is parallel to the third side.

§ **271.** In any triangle, the bisector of an interior angle divides the opposite side internally into segments proportional to the adjacent sides of the triangle.

§ **273.** Two polygons are similar if their corresponding angles are equal and their corresponding sides are proportional.

§ **275.** Two triangles are similar if they are mutually equiangular.

§ **276.** Two triangles are similar if two angles of one are equal respectively to two angles of the other.

§ **277.** Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

§ **280.** Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.

§ **281.** Two triangles are similar if their corresponding sides are proportional.

§ **282.** Corresponding altitudes of similar triangles have the same ratio as any two corresponding sides.

§ **283.** If two chords are drawn through a fixed point within a circle, the product of the segments of one is equal to the product of the segments of the other.

§ **285.** If any two secants are drawn through a fixed point outside a circle, the product of one and its external segment equals the product of the other and its external segment.

§ **287.** If a secant and a tangent are drawn to a circle from the same outside point, the square of the tangent equals the product of the whole secant and its external segment.

§ **288.** If the altitude be drawn to the hypotenuse of a right triangle:

I. The altitude is the mean proportional between the segments of the hypotenuse.

II. Each leg is the mean proportional between the whole hypotenuse and the adjacent segment.

§ **290.** If a perpendicular be drawn to the diameter of a circle from any point on the circle:

(*a*) it is the mean proportional between the segments of the diameter; and

(*b*) the chord drawn from the point to one end of the diameter is the mean proportional between the whole diameter and that segment of it which is adjacent to the chord.

§ 291. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.

§ 292. The square of either leg of a right triangle equals the square of the hypotenuse minus the square of the other leg.

§ 293. Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.

§ 295. Two similar polygons can be separated into the same number of triangles, similar each to each, and similarly placed.

§ 296. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

§ 297. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

§ 307. In any triangle, the bisector of an exterior angle at any vertex divides the opposite side externally into two segments whose ratio equals the ratio of the two adjacent sides of the triangle.

§ 311. In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other upon it.

§ 312. In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.

§ 314. When the three sides of a triangle are a, b, and c, the altitude to side a is

$$\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$.

§ 315. In any triangle, the sum of the squares of two sides equals twice the square of half the third side plus twice the square of the median drawn to that side.

§ 316. The difference between the squares of two sides of a triangle equals twice the product of the third side and the projection of the median upon that side.

§ 317. In any triangle, the product of two sides equals the product of the diameter of the circumscribed circle and the altitude upon the third side.

§ 319. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.

§ **324.** Two congruent figures are necessarily equal, but two equal figures are not necessarily congruent.

§ **326.** The area of a rectangle is the product of its base and its altitude.

§ **327.** Comparison of rectangles.

(*a*) Two rectangles having equal bases and equal altitudes are equal.

(*b*) Two rectangles are to each other as the products of their altitudes and their bases.

(*c*) Two rectangles having equal bases are to each other as their altitudes.

(*d*) Two rectangles having equal altitudes are to each other as their bases.

§ **328.** The area of a parallelogram equals the product of its base and altitude.

§ **329.** **I.** Parallelograms having equal bases and equal altitudes are equal.

II. Two parallelograms are to each other as the products of their bases and their altitudes.

III. Parallelograms having equal altitudes are to each other as their bases.

IV. Parallelograms having equal bases are to each other as their altitudes.

§ **330.** The area of a triangle equals one half the product of its base and altitude.

§ **331.** **I.** Triangles having equal bases and equal altitudes are equal.

II. Two triangles are to each other as the products of their bases and their altitudes.

III. Triangles having equal altitudes are to each other as their bases.

IV. Triangles having equal bases are to each other as their altitudes.

V. A triangle is one half a parallelogram having the same base and altitude.

§ **333.** The area of a trapezoid equals one half its altitude multiplied by the sum of its bases.

§ **334.** The area of a trapezoid equals the product of its altitude and its median.

§ **335.** The areas of two similar triangles are to each other as the squares of any two corresponding sides.

§ **336.** The areas of two similar polygons are to each other as the squares of any two corresponding sides.

§ **337.** The square upon the hypotenuse of a right triangle is equal to the sum of the squares upon the two legs of the triangle.

§ **341.** Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including these angles.

§ **349.** Each angle of a regular polygon having n sides is $\left(\frac{n-2}{n}\right)$ straight angles.

§ **350.** A circle can be circumscribed about any regular polygon.

§ **351.** A circle can be inscribed in any regular polygon.

§ **353.** The central angle of a regular n-gon is $\frac{360^\circ}{n}$.

§ **354.** If a circle be divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon of that number of sides.

§ **355.** If from the mid-point of each arc subtended by a side of a regular polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.

§ **356.** An equilateral polygon inscribed in a circle is regular.

§ **357.** If a circle be divided into any number of equal arcs, the tangents at the points of division form a regular circumscribed polygon of that number of sides.

§ **358.** Tangents drawn to the circle at the mid-points of the arcs included between two consecutive points of contact of a regular circumscribed polygon form, with the sides of the original circumscribed polygon, a regular circumscribed polygon having double the number of sides.

§ **359.** The area of a regular polygon is equal to one half the product of its apothem and its perimeter.

§ **366.** Regular polygons of the same number of sides are similar.

§ **367.** The perimeters of two regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.

§ **368.** The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.

§ **370.** (*a*) A variable is a number which assumes different values during a particular discussion.

(*b*) A constant is a number which has a fixed value during a particular discussion.

(*c*) The limit of a variable is a constant such that the numerical value of the difference between it and the variable becomes and remains less than any small positive number.

The symbol for " approaches the limit " is $\longrightarrow$.

§ **371.** (*a*) If a variable x approaches a limit l, then cx, where c is a constant, approaches the limit cl.

(*b*) If two variables are constantly equal, and each approaches a limit, their limits are equal.

§ **372.** We define:

(*a*) The length of a circle is the limit which the perimeters of regular inscribed polygons approach when the number of their sides is increased indefinitely.

(*b*) The area of a circle is the limit which the areas of regular inscribed polygons approach when the number of their sides is increased indefinitely.

§ **373.** If regular polygons be inscribed in a circle, the apothems of the polygons approach the radius of the circle as limit as the number of the sides of the polygons increases indefinitely.

§ **374.** The circumferences of two circles have the same ratio as their radii or their diameters.

§ **375.** The ratio of the circumference of one circle to its diameter equals the ratio of the circumference of any other circle to its diameter.

§ **376.** The constant ratio of a circumference of a circle to its diameter is denoted by the Greek letter π.

§ **378.** The area of a circle is the product of one half its radius and its circumference.

§ **379.** The area of a circle of radius $r = \pi r^2$.

§ **381.** The areas of two circles have the same ratio as the squares of their radii, or the squares of their diameters.

§ **383.** The area of a sector is to the area of the circle as its angle is to 360°.

FORMULAS FROM PLANE GEOMETRY

I. Formula for the altitude of a triangle whose sides are a, b, and c, and whose perimeter is $2s$:

altitude to side a = $h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$.

The altitude of an equilateral triangle whose side is b:

$$h_b = \frac{b}{2}\sqrt{3}.$$

II. Formulas for areas of plane figures:

Let a = the number of linear units in the altitude;
b = the number of the same kind of units in the base;
A = the number of the corresponding surface units in the area.
Let r = the number of linear units in the radius.
Let p = the number of linear units in the perimeter.

FOR THE AREA OF	THE FORMULA IS
1. A rectangle.	**1.** $A = ab$.
2. A parallelogram.	**2.** $A = ab$.
3. A triangle.	**3.** $A = \frac{1}{2}ab$.
4. A trapezoid.	**4.** $A = \frac{1}{2}a(b_1 + b_2)$.
5. A circle.	**5.** $A = \pi r^2$, or $A = \frac{1}{4}\pi d^2$.
6. A regular polygon.	**6.** $A = \frac{1}{2}$ apothem $\times p$.
7. An equilateral triangle.	**7.** $A = \frac{b^2}{4}\sqrt{3}$, where b = one side.
8. Any triangle.	**8.** $A = \sqrt{s(s-a)(s-b)(s-c)}$.
9. A sector of a circle.	**9.** $A = \frac{n}{360} \times \pi r^2$, where n = the number of degrees in the $\angle$ of the sector.

III. **1.** The circumference of a circle = $2\pi r$; or πd.

2. The length of an arc of a circle = $\frac{n}{360} \times 2\pi r$, where n = the number of degrees in the arc.

FORMULAS FROM SOLID GEOMETRY

NOTATION

h = length of altitude.
s = length of slant height.
e = length of edge or element.
p = perimeter of lower base.
p' = perimeter of upper base.
m = perimeter of mid-section.
r = radius of lower base.
r' = radius of upper base.

c = circumference of lower base.
c' = circumference of upper base.
A = lateral area.
T = total area.
B = area of lower base.
B' = area of upper base.
M = area of mid-section.
V = volume.

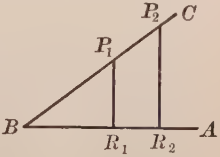
FIGURE	FORMULA FOR THE AREA	FORMULA FOR THE VOLUME
Rectangular parallelepiped	....	V = product of the three dimensions
Oblique parallelepiped	....	$V = Bh$
Any prism	$A = p$(of rt. sect.)$\times e$	$V = Bh$
Right prism	$A = pe$; or $A = ph$	$V = Bh$
Any pyramid	....	$V = \frac{1}{3}Bh$
Regular pyramid	$A = \frac{1}{2}sp$	$V = \frac{1}{3}Bh$
Frustum of any pyramid	....	$V = \frac{1}{3}h(B + B' + \sqrt{BB'})$
Frustum of a regular pyramid	$A = \frac{1}{2}s(p + p')$; $A = sm$	$V = \frac{1}{3}h(B + B' + \sqrt{BB'})$
Prismatoid	....	$V = \frac{1}{6}h(B + B' + 4M)$
Any cylinder	$A = p$(of rt. sect.)$\times e$	$V = Bh$
Right circular cylinder	$A = ch$; or $A = 2\pi rh$ $T = 2\pi r(r + h)$	$V = \pi r^2 h$
Any cone	....	$V = \frac{1}{3}Bh$
Right circular cone	$A = \frac{1}{2}cs$; $A = \pi rs$; $T = \pi r(r + s)$	$V = \frac{1}{3}\pi r^2 h$
Frustum of rt. circular cone	$A = \frac{1}{2}s(c + c')$; or $A = \pi s(r + r')$	$V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$
Sphere	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$
Lune	$A = \frac{\pi r^2 a}{90}\cdot\left(\begin{array}{l}a = \text{no. of}\\ \text{degrees in}\\ \angle \text{ of lune.}\end{array}\right)$	....
Zone	$A = 2\pi rh$	....
Spherical triangle	$A = \frac{\pi r^2 E}{180}\cdot\left(\begin{array}{l}E = \text{no. of}\\ \text{degrees in}\\ \text{excess of}\\ \triangle.\end{array}\right)$	....

ELEMENTARY TRIGONOMETRY

1. Sine of an angle. Let $\angle ABC$ be any angle. On BC, take points P_1 and P_2. Draw perpendiculars P_1R_1 and P_2R_2 to AB.

Then $\triangle BP_1R_1 \sim \triangle BP_2R_2$.

$$\frac{P_1R_1}{P_1B} = \frac{P_2R_2}{P_2B}$$



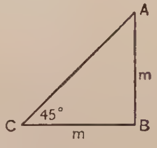
That is, *the ratio of the perpendicular* PR *to the distance* PB *is the same, wherever* P *may be located on* BC.

This *constant ratio* is called the **sine of ∠ B.** **(Sin B.)**

When the angle is acute, its sides and the perpendicular form a right triangle. *In this right triangle,*

the sine of an (acute) angle = opposite side ÷ hypotenuse.

2. The approximate value of the sines of certain angles can be computed by elementary means.



Example 1. Determine the sine of 45°.

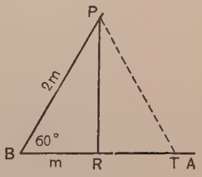
Solution. **1.** Let $\angle C = 45°$. Draw $AB \perp CB$.

2. Then $AB = CB$, and, if $CB = m$, $AB = m$.

3. $\therefore CA^2 = m^2 + m^2$; $CA^2 = 2\,m^2$; $CA = m\sqrt{2}$.

4. $\therefore \sin C = \dfrac{AB}{AC} = \dfrac{m}{m\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2\sqrt{2}} = \dfrac{\sqrt{2}}{2} = \dfrac{1.414}{2} = .707.$

Example 2. Determine the sine of 60°.



Solution. **1.** Let $\angle B = 60°$. Draw $PR \perp BA$. Take $PT = PB$. Then $\angle T = 60°$.

2. $\triangle PBT$ is equilateral, and $BR = \frac{1}{2}\,BP$.

3. Let $BP = 2\,m$. Then $BR = m$.

4. In $\triangle BPR$, $PR^2 = (2\,m)^2 - m^2$, or $3\,m^2$.

5. $\therefore PR = \sqrt{3\,m^2}$, or $PR = m\sqrt{3}$.

6. $\therefore \sin 60° = \dfrac{PR}{PB} = \dfrac{m\sqrt{3}}{2\,m} = \dfrac{1.732}{2} = .866.$

Example 3. Find the sine of 30°.

Solution. 1. In the figure of Example 2, $\sin 30° = \frac{BR}{BP}$.

2. $\therefore \sin 30° = \frac{m}{2m} = \frac{1}{2} = .500.$

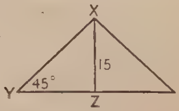
(*a*) When an angle changes, its sine changes; or *the sine of an angle is a function of the angle.*

Observe: $\sin 30° = .500$; $\sin 45° = .707$; $\sin 60° = .866$.

(*b*) *When an angle increases, its sine increases.*

3. Use of the sine of an angle.

Example. How long must the rafter XY be cut if the rise XZ is 15′ and $\angle Y$ is 45°?



Solution. 1. $\text{Sin } 45° = \frac{XZ}{XY}$, or $\sin 45° \cdot XY = 15$.

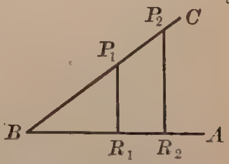
2. $\therefore XY = \frac{15}{\sin 45} = \frac{15}{.707} = 21.2$ ft.

To make practical use of the sine, it is necessary to have the sines of a large number of angles. These values are given in the table on pages 496 to 500.

Thus, on page 496, the sine of 8° 40′ = .1507.

By means of this table, the approximate values of the sines of other angles can be computed. This will be explained later.

4. The cosine of an angle.



In the adjoining figure,

$$\frac{BR_1}{BP_1} = \frac{BR_2}{BP_2}.$$

That is, for all positions of P, the ratio $\frac{BR}{BP}$ is constant.

This ratio is **the cosine of the angle.** **(Cos B.)**

In the right triangle formed when the angle is acute:

cosine of an acute angle = adjacent side ÷ hypotenuse.

Using the results of Examples 1–3 of § 2:

$$\text{Cos } 30° = \frac{PR}{BP} = \frac{m\sqrt{3}}{2\,m} = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866.$$

$$\text{Cos } 45° = \frac{CB}{AC} = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.414}{2} = .707.$$

$$\text{Cos } 60° = \frac{BR}{BP} = \frac{m}{2\,m} = \frac{1}{2} = .500.$$

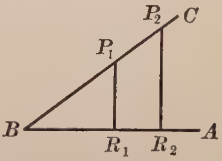
(*a*) When an angle changes, its cosine changes; *i.e. the cosine of an angle is a function of the angle.*

(*b*) *When an angle increases, its cosine decreases.*

(*c*) The cosine of many angles, correct to four decimal places, are given in the table on pages 496 to 500.

Thus, cos 38° 20′ = .7844.

5. The tangent of an angle.



In the adjoining figure

$$\frac{P_1R_1}{BR_1} = \frac{P_2R_2}{BR_2}.$$

That is, for all positions of P on BC, the ratio $\frac{PR}{BR}$ is constant. This ratio is called the **tangent of angle B. (Tan B.)**

In the right triangle formed when the angle is acute, the *tangent of acute angle = opposite side ÷ adjacent side.*

Using the results in Examples 1–3, § 2:

$$\text{Tan } 30° = \frac{BR}{PR} = \frac{m}{m\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.732}{3} = .577.$$

$$\text{Tan } 45° = \frac{AB}{CB} = \frac{m}{m} = 1.$$

$$\text{Tan } 60° = \frac{PR}{BR} = \frac{m\sqrt{3}}{m} = \sqrt{3} = 1.732.$$

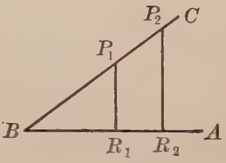
(*a*) When an angle changes, its tangent changes; *i.e., the tangent of an angle is a function of the angle.*

(*b*) *When an angle increases, its tangent increases.*

(*c*) The tangents of many angles, correct to four decimal places, are given in the table on pages 496 to 500.

Thus, tan 43° 10′ = .9380.

6. The cotangent of an angle.



In the adjoining figure,

$$\frac{BR_1}{P_1R_1} = \frac{BR_2}{P_2R_2}.$$

That is, for all positions of P on BC, the ratio $\frac{BR}{PR}$ is constant.

This ratio is called the **cotangent of angle B. (Cot B)** In the right triangle formed when the angle is acute,

cotangent of acute ∠ = *adjacent side* ÷ *opposite side.*

Using the results in Examples 1–3, § 2,

$$\cot 30^\circ = \frac{PR}{BR} = \frac{m\sqrt{3}}{m} = \sqrt{3} = 1.732.$$

$$\cot 45^\circ = \frac{CB}{AB} = \frac{m}{m} = 1.$$

$$\cot 60^\circ = \frac{BR}{PR} = \frac{m}{m\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1.732}{3} = .577.$$

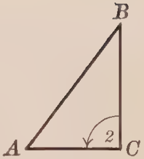
(*a*) When an angle changes, its cotangent changes; *i.e., the cotangent of an angle is a function of the angle.*

(*b*) *When an angle increases, its cotangent decreases.*

(*c*) Cotangents of many angles, correct to 4 decimal places, are given in the table on pages 496 to 500.

Thus, cot 18° 50′ = 2.9319.

7. Cofunctions.



In the adjoining figure, observe that:

(*a*) $\sin A = \frac{BC}{AB}$, and $\cos B = \frac{BC}{AB}$.

But $B = 90 - A$.

$\therefore \sin A = \cos(90 - A)$.

(*b*) $\cos A = \frac{AC}{AB}$, and $\sin B = \frac{AC}{AB}$.

$\therefore \cos A = \sin(90 - A)$.

(*c*) $\tan A = \frac{BC}{AC}$, and $\cot B = \frac{BC}{AC}$.

$\therefore \tan A = \cot(90 - A)$.

The sine and cosine are called cofunctions of *each other; i.e.*, the sine is the cofunction of the cosine, as well as the cosine is the cofunction of the sine.

Similarly the tangent and cotangent are cofunctions of each other.

As a consequence of parts *a*, *b*, and *c*, and the definitions, it appears that

Any function of an angle equals the cofunction of the complementary angle.

8. The fact developed in § **7** makes it possible to find the value of any function of an angle from 45° to 90° in a table of the values of the cofunction of the angles from 0° to 45°.

Thus,
$$\cos 84° = \sin 6° = .1045.$$
$$\sin 82° 30' = \cos 7° 30' = .9914.$$

For this reason the table on pages 496 to 500 consists primarily of the values of the functions from 0° to 45°. Observe its second column headed Sin. The numbers in it are the sines of the angles opposite them at the left in column one. But at the foot of column two is the abbreviation Cos. This means that the numbers in the column are also the cosines of certain angles, namely the angles opposite them *at the right* in column six.

Thus, $\sin 6^\circ 10' = \cos 83^\circ 50' = .1074.$

To find the value of a function of an angle greater than 45°, which is listed in the table, find the angle in column six; find the desired function *at the bottom of the page;* then above that function, find the value opposite the given angle.

Thus, $\cos 67^\circ 40' = .3800$ (On page 498.)

9. Practice in using the trigonometric table.

(*a*) Recall first the facts developed in part *b* of § 2, § 4, and § 5.

(*b*) The approximate value of a function of any angle from 0° to 90° may be found from the table.

Example 1. Find sin 27° 25′.

Solution. **1.**

$$\left.\begin{aligned} \sin 27^\circ 20' &= .4592 \\ \sin 27^\circ 25' &= \quad ? \\ \sin 27^\circ 30' &= .4617 \end{aligned}\right\} \text{difference} = .0025.$$

2. 5′ is $\frac{1}{2}$ of 10′. $\frac{1}{2}$ of $.0025 = .0013$.

3. Since the sine increases as the angle increases,

$$\sin 27^\circ 25' = .4592 + .0013, \text{ or } .4605.$$

Example 2. Find cos 62° 18′.

Solution. **1.**

$$\left.\begin{aligned} \cos 62^\circ 10' &= .4669 \\ \cos 62^\circ 18' &= \quad ? \\ \cos 62^\circ 20' &= .4643 \end{aligned}\right\} \text{difference} = .0026.$$

2. $8' = \frac{8}{10}$ of 10′. $.8 \times .0026 = .00208$, or .0021.

3. Since the cosine decreases as the angle increases,

$$\cos 62^\circ 18' = .4669 - .0021, \text{ or } .4648.$$

Example 3. Find tan 72° 33′.

Solution. **1.**

$$\left.\begin{aligned} \tan 72^\circ 30' &= 3.1716 \\ \tan 72^\circ 33' &= \quad ? \\ \tan 72^\circ 40' &= 3.2041 \end{aligned}\right\} \text{difference} = .0325.$$

2. 3′ is $\frac{3}{10}$ of 10′. $.3 \times .0325 = .00975$, or .0098.

3. Since the tangent increases as the angle increases,

$$\tan 72^\circ 33' = 3.1716 + .0098, \text{ or } 3.1814.$$

Example 4. What is the angle x when $\sin x = .6354$?

Solution. 1.
$$\left.\begin{array}{ll} \left.\begin{array}{l}\sin 39^\circ 20' = .6338 \\ \sin x \qquad\; = .6354\end{array}\right\} & \begin{array}{l}\text{difference}\\ = .0016\end{array} \\ \sin 39^\circ 30' = .6361 & \end{array}\right\} \text{difference} = .0023.$$

2. $.0016 \div .0023 = .69.$ $\quad .69 \times 10' = 6.9$, or $7'$.

3. Since the sine increases when the angle increases,
$$x = 39^\circ 20' + 7' \text{ or } 39^\circ 27'.$$

Example 5. What is the angle x when $\cos x = .5874$?

Solution. 1.
$$\left.\begin{array}{ll} \left.\begin{array}{l}\cos 54^\circ\ \ 0' = .5878 \\ \cos x \qquad = .5874\end{array}\right\} & \begin{array}{l}\text{difference}\\ = .0004\end{array} \\ \cos 54^\circ 10' = .5854 & \end{array}\right\} \text{difference} = .0024.$$

2. $.0004 \div .0024 = \frac{1}{6}.$ $\quad \frac{1}{6} \times 10' = 1\frac{2}{3}$, or $2'$.

3. Since the cosine decreases as the angle increases,
$$x = 54^\circ 0' + 2', \text{ or } 54^\circ 2'.$$

Ex. 1. Find: $\sin 40^\circ$; $\sin 70^\circ$; $\sin 38^\circ 40'$.

Ex. 2. Find $\cos 37^\circ$; $\cos 65^\circ$; $\cos 54^\circ 50'$.

Ex. 3. Find $\tan 18^\circ$; $\tan 32^\circ 50'$; $\tan 80^\circ 20'$.

Ex. 4. Find x if: $\sin x = .4683$; $\sin x = .7538$.

Ex. 5. Find x if: $\cos x = .8592$; $\cos x = .2476$.

Ex. 6. Find x if: $\tan x = .6775$; $\tan x = 1.8321$.

Ex. 7. Find x if: $\cos x = .3795$; $\cos x = .6594$.

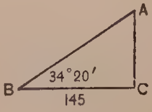
10. Use of the trigonometric functions in plane figures.

Example 1. AC represents a pole of unknown height. If B is 145 ft. from its foot C, and if the angle of elevation of A at B (namely $\angle CBA$) is 34° 20′, find AC.

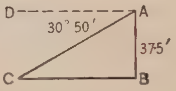
Solution. 1. The function of $\angle B$ connecting AC and BC is the tangent. That is,
$$\tan B = \frac{AC}{BC}, \text{ or } \tan 34^\circ 20' = \frac{AC}{145}.$$

2. $AC = 145 \times \tan 34^\circ 20'.$

3. $AC = 145 \times .6830$, or 99.035 ft.
That is, AC is about 99 ft. high.



Example 2. AB represents the height of a hill 375 ft. above the level of a lake. AD is an imaginary line parallel to BC. C represents the position of a ship. $\angle DAC = 30° 50'$. Find AC.



Solution. 1. $\angle BCA = \angle DAC = 30° 50'$.

2. The function of $\angle C$ connecting AB and AC is the sine of $\angle C$; or $\sin C = \frac{AB}{AC}$.

$$\sin 30° 50' = \frac{375}{AC}, \text{ or } AC = \frac{375}{\sin 30° 50'}$$

3. $$AC = \frac{375}{.5125} = 731.7 \text{ ft.}$$

Note 1. $\angle DAC$ is the **angle of depression** of C at A.

Note 2. If the pupils know how to use logarithms, the following solution is possible:

1. $$AC = \frac{375}{\sin 30° 50'}$$

2. $$\log AC = \begin{array}{rl} \log 375 &= 2.5740 \\ -\log \sin 30° 50' &= \underline{9.7097 - 10} \end{array} \qquad \text{(See p. 504)}$$

$$\therefore \log AC = 2.8643.$$

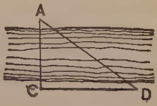
3. $$AC = 731.66, \text{ or } 731.7.$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \log 731 = 2.8639 \\ \log AC = 2.8643 \end{array}\right\} \text{Diff.} = .0004 \\ \log 732 = 2.8645 \end{array}\right\} \text{Diff.} = .0006$$

$$.0004 \div .0006 = \tfrac{2}{3} = .66.$$

EXAMPLES FROM PLANE GEOMETRY

Ex. 8. In the adjoining figure, AC is perpendicular to CD; $CD = 250$ ft.; angle $D = 42° 30'$. Find AC.



Ex. 9. At a point 185 feet from the foot of a high building the angle of elevation of the top is 37° 20′. How high is the building?

Ex. 10. From the top of a hill known to be 275 ft. above the level of the plain, the angle of depression of a house is 32° 45′. How far away is the house from an imaginary point directly below the top of the hill?

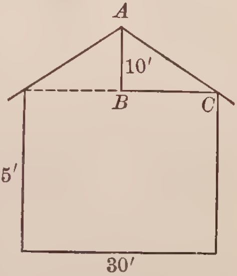
Ex. 11. In $\triangle ABC$, BC is 20 ft., $\angle B$ is 65° 40′, and AB is 8 ft. Draw a triangle to represent these conditions and draw the altitude AD.

a. Find the length of AD.

b. Having found AD, find the area.

Ex. 12. In the figure at the right is a right triangle in which $AB = 10$ ft. and $BC = 20'$.

a. Find $\angle C$. *b.* Find AC.



Ex. 13. At a point 215 ft. from the foot of a high building, the angle of elevation of the top is 37° 50′. How high is the building?

Ex. 14. From a height of 350 ft., the angle of depression of an object on the plain below is 62° 25′. Find the distance of the object from a point in the plain directly below the point of observation.

Ex. 15. In $\triangle ABC$, in which $\angle C$ is a right angle, and $\angle B$ is an angle of 34° 45′, side AB is 22 in. long. Find the length of AC and of BC.

Ex. 16. The angle of elevation of an airplane at a certain point P is 58° 30′. Point D, 1500 ft. distant, is directly below the airplane. How high is the airplane?

Ex. 17. From a hilltop 145 feet above the level of a lake, the angle of depression of one sailboat is 32° 40′. The angle of depression of a second boat directly in line with the first boat is 48° 26′. What is the distance between the two boats?

Ex. 18. An observer in an airplane, which is 2800 feet high, finds that the angle of depression of a station on the ground is 35° 18′. How far is it from his present position to a point directly over the station?

Ex. 19. If the radius of a circle is 4 in., what is the length of a chord which subtends an arc of 38°?

Ex. 20. If a circle of radius 15 in. be divided into five equal arcs, what is the length of the chord subtending one of the arcs?

Ex. 21. The mid-point of a chord 12 in. long is 5 in. from the mid-point of the arc subtended by the chord. How many degrees are there in the arc subtended by the chord?

Ex. 22. In rt. $\triangle ABC$, having $\angle A = 90°$, $\angle B = 25°$, and $AB = 25$ in., how long are: AC? BC? the altitude AD to BC?

Ex. 23. In $\triangle ABC$, having $AB = AC$, and altitude AD drawn to base BC, find $\angle A$ and side BC if $AB = 35$ in. and $AD = 18$ in.

Ex. 24. At a point 200 ft. from the foot of a tower, on which stands a flagpole, the angle of elevation of the top of the tower is 35°, and of the top of the pole 47°. How high is the tower, and how long is the pole?

Ex. 25. How long is the radius of the circle which is inscribed in an equilateral triangle whose sides are 15 in. long?

Ex. 26. From a point 200 feet from the foot of a tower surmounted by a flagpole, the angle of elevation of the top of the pole is $39° \, 18'$; from a point 150 ft. further away in a straight line with the tower, the angle of elevation of the foot of the pole is 22°. How long is the pole?

Ex. 27. Prove that the area of $\triangle ABC$ whose sides are a, b, and c, equals $\frac{1}{2} ac \sin B$.

EXAMPLES FROM SOLID GEOMETRY

Ex. 28. The altitude of a right prism is 15 in. Its base is inclosed by a regular pentagon, one of whose sides is 4 in. long. Find its volume.

Ex. 29. In triangular pyramid $A\text{–}BCD$, BC is 15 in., CD is 18 in., and $\angle C$ is $50° \, 20'$. The altitude from A to BCD is 20 in. What is the volume of $A\text{–}BCD$?

Ex. 30. What is the lateral area of the regular pyramid whose base is inclosed by an equilateral triangle whose sides are 15 in. long, if the lateral edges are 20 in. long?

Ex. 31. What is the volume of the pyramid in Exercise 30?

Ex. 32. What is the lateral area and the volume of the regular tetraedron whose edges are 24 in. long?

Ex. 33. What is the volume of the right circular cone whose slant height is 25 in. and whose base radius is 5 in.?

Ex. 34. What is the lateral area and the volume of the right circular cone whose altitude is 18 in. and whose slant height is 22 in.?

Ex. 35. What is the volume of the oblique parallelepiped whose altitude is 8 in., and whose base is parallelogram $ABCD$ having $AB = 4$ in., $\angle B = 58°$, and $BC = 10$ in.?

TABLE I. CERTAIN POWERS AND ROOTS

No.	Sqs.	Sq. Roots	Cubes	Cube Roots	No.	Sqs.	Sq. Roots	Cubes	Cube Roots
1	1	1.000	1	1.000	**51**	2,601	7.141	132,651	3.708
2	4	1.414	8	1.259	**52**	2,704	7.211	140,608	3.732
3	9	1.732	27	1.442	**53**	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	**54**	2,916	7.348	157,464	3.779
5	25	2.236	125	1.709	**55**	3,025	7.416	166,375	3.802
6	36	2.449	216	1.817	**56**	3,136	7.483	175,616	3.825
7	49	2.645	343	1.912	**57**	3,249	7.549	185,193	3.848
8	64	2.828	512	2.000	**58**	3,364	7.615	195,112	3.870
9	81	3.000	729	2.080	**59**	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	**60**	3,600	7.745	216,000	3.914
11	121	3.316	1,331	2.223	**61**	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	**62**	3,844	7.874	238,328	3.957
13	169	3.605	2,197	2.351	**63**	3,969	7.937	250,047	3.979
14	196	3.741	2,744	2.410	**64**	4,096	8.000	262,144	4.000
15	225	3.872	3,375	2.466	**65**	4,225	8.062	274,625	4.020
16	256	4.000	4,096	2.519	**66**	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	**67**	4,489	8.185	300,763	4.061
18	324	4.242	5,832	2.620	**68**	4,624	8.246	314,432	4.081
19	361	4.358	6,859	2.668	**69**	4,761	8.306	328,509	4.101
20	400	4.472	8,000	2.714	**70**	4,900	8.366	343,000	4.121
21	441	4.582	9,261	2.758	**71**	5,041	8.426	357,911	4.140
22	484	4.690	10,648	2.802	**72**	5,184	8.485	373,248	4.160
23	529	4.795	12,167	2.843	**73**	5,329	8.544	389,017	4.179
24	576	4.898	13,824	2.884	**74**	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	**75**	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	**76**	5,776	8.717	438,976	4.235
27	729	5.196	19,683	3.000	**77**	5,929	8.774	456,533	4.254
28	784	5.291	21,952	3.036	**78**	6,084	8.831	474,552	4.272
29	841	5.385	24,389	3.072	**79**	6,241	8.888	493,039	4.290
30	900	5.477	27,000	3.107	**80**	6,400	8.944	512,000	4.308
31	961	5.567	29,791	3.141	**81**	6,561	9.000	531,441	4.326
32	1,024	5.656	32,768	3.174	**82**	6,724	9.055	551,368	4.344
33	1,089	5.744	35,937	3.207	**83**	6,889	9.110	571,787	4.362
34	1,156	5.830	39,304	3.239	**84**	7,056	9.165	592,704	4.379
35	1,225	5.916	42,875	3.271	**85**	7,225	9.219	614,125	4.396
36	1,296	6.000	46,656	3.301	**86**	7,396	9.273	636,056	4.414
37	1,369	6.082	50,653	3.332	**87**	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.361	**88**	7,744	9.380	681,472	4.447
39	1,521	6.245	59,319	3.391	**89**	7,921	9.433	704,969	4.464
40	1,600	6.324	64,000	3.419	**90**	8,100	9.486	729,000	4.481
41	1,681	6.403	68,921	3.448	**91**	8,281	9.539	753,571	4.497
42	1,764	6.480	74,088	3.476	**92**	8,464	9.591	778,688	4.514
43	1,849	6.557	79,507	3.503	**93**	8,649	9.643	804,357	4.530
44	1,936	6.633	85,184	3.530	**94**	8,836	9.695	830,584	4.546
45	2,025	6.708	91,125	3.556	**95**	9,025	9.746	857,375	4.562
46	2,116	6.782	97,336	3.583	**96**	9,216	9.797	884,736	4.578
47	2,209	6.855	103,823	3.608	**97**	9,409	9.848	912,673	4.594
48	2,304	6.928	110,592	3.634	**98**	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	**99**	9,801	9.949	970,299	4.626
50	2,500	7.071	125,000	3.684	**100**	10,000	10.000	1,000,000	4.641

TABLE II. LOGARITHMS OF NUMBERS

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

TABLE II (*Continued*)

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

TABLE III. Values of the Sines, Cosines, Tangents, and Cotangents of Certain Angles

Angle	Sin	Cos	Tan	Cot	Angle
0° 0′	.0000	1.0000	.0000		90° 0′
10′	.0029	1.0000	.0029	343.77	89° 50′
20′	.0058	1.0000	.0058	171.89	40′
30′	.0087	1.0000	.0087	114.59	30′
40′	.0116	.9999	.0116	85.940	20′
50′	.0145	.9999	.0145	68.750	89° 10′
1° 0′	.0175	.9998	.0175	57.290	89° 0′
10′	.0204	.9998	.0204	49.104	88° 50′
20′	.0233	.9997	.0233	42.964	40′
30′	.0262	.9997	.0262	38.188	30′
40′	.0291	.9996	.0291	34.368	20′
50′	.0320	.9995	.0320	31.242	88° 10′
2° 0′	.0349	.9994	.0349	28.636	88° 0′
10′	.0378	.9993	.0378	26.432	87° 50′
20′	.0407	.9992	.0407	24.542	40′
30′	.0436	.9990	.0437	22.904	30′
40′	.0465	.9989	.0466	21.470	20′
50′	.0494	.9988	.0495	20.206	87° 10′
3° 0′	.0523	.9986	.0524	19.081	87° 0′
10′	.0552	.9985	.0553	18.075	86° 50′
20′	.0581	.9983	.0582	17.169	40′
30′	.0610	.9981	.0612	16.350	30′
40′	.0640	.9980	.0641	15.605	20′
50′	.0669	.9978	.0670	14.924	86° 10′
4° 0′	.0698	.9976	.0699	14.301	86° 0′
10′	.0727	.9974	.0729	13.727	85° 50′
20′	.0756	.9971	.0758	13.197	40′
30′	.0785	.9969	.0787	12.706	30′
40′	.0814	.9967	.0816	12.251	20′
50′	.0843	.9964	.0846	11.826	85° 10′
5° 0′	.0872	.9962	.0875	11.430	85° 0′
10′	.0901	.9959	.0904	11.059	84° 50′
20′	.0929	.9957	.0934	10.712	40′
30′	.0958	.9954	.0963	10.385	30′
40′	.0987	.9951	.0992	10.078	20′
50′	.1016	.9948	.1022	9.7882	84° 10′
6° 0′	.1045	.9945	.1051	9.5144	84° 0′
10′	.1074	.9942	.1080	9.2553	83° 50′
20′	.1103	.9939	.1110	9.0098	40′
30′	.1132	.9936	.1139	8.7769	30′
40′	.1161	.9932	.1169	8.5555	20′
50′	.1190	.9929	.1198	8.3450	83° 10′
7° 0′	.1219	.9925	.1228	8.1443	83° 0′
10′	.1248	.9922	.1257	7.9530	82° 50′
20′	.1276	.9918	.1287	7.7704	40′
30′	.1305	.9914	.1317	7.5958	30′
40′	.1334	.9911	.1346	7.4287	20′
50′	.1363	.9907	.1376	7.2687	82° 10′
8° 0′	.1392	.9903	.1405	7.1154	82° 0′
10′	.1421	.9899	.1435	6.9682	81° 50′
20′	.1449	.9894	.1465	6.8269	40′
30′	.1478	.9890	.1495	6.6912	30′
40′	.1507	.9886	.1524	6.5606	20′
50′	.1536	.9881	.1554	6.4348	81° 10′
9° 0′	.1564	.9877	.1584	6.3138	81° 0′
Angle	Cos	Sin	Cot	Tan	Angle

TABLE III (*Continued*)

ANGLE	SIN	COS	TAN	COT	ANGLE
9° 0′	.1564	.9877	.1584	6.3138	**81° 0′**
10′	.1593	.9872	.1614	6.1970	**80° 50′**
20′	.1622	.9868	.1644	6.0844	**40′**
30′	.1650	.9863	.1673	5.9758	**30′**
40′	.1679	.9858	.1703	5.8708	**20′**
50′	.1708	.9853	.1733	5.7694	**80° 10′**
10° 0′	.1736	.9848	.1763	5.6713	**80° 0′**
10′	.1765	.9843	.1793	5.5764	**79° 50′**
20′	.1794	.9838	.1823	5.4845	**40′**
30′	.1822	.9833	.1853	5.3955	**30′**
40′	.1851	.9827	.1883	5.3093	**20′**
50′	.1880	.9822	.1914	5.2257	**79° 10′**
11° 0′	.1908	.9816	.1944	5.1446	**79° 0′**
10′	.1937	.9811	.1974	5.0658	**78° 50′**
20′	.1965	.9805	.2004	4.9894	**40′**
30′	.1994	.9799	.2035	4.9152	**30′**
40′	.2022	.9793	.2065	4.8430	**20′**
50′	.2051	.9787	.2095	4.7729	**78° 10′**
12° 0′	.2079	.9781	.2126	4.7046	**78° 0′**
10′	.2108	.9775	.2156	4.6382	**77° 50′**
20′	.2136	.9769	.2186	4.5736	**40′**
30′	.2164	.9763	.2217	4.5107	**30′**
40′	.2193	.9757	.2247	4.4494	**20′**
50′	.2221	.9750	.2278	4.3897	**77° 10′**
13° 0′	.2250	.9744	.2309	4.3315	**77° 0′**
10′	.2278	.9737	.2339	4.2747	**76° 50′**
20′	.2306	.9730	.2370	4.2193	**40′**
30′	.2334	.9724	.2401	4.1653	**30′**
40′	.2363	.9717	.2432	4.1126	**20′**
50′	.2391	.9710	.2462	4.0611	**76° 10′**
14° 0′	.2419	.9703	.2493	4.0108	**76° 0′**
10′	.2447	.9696	.2524	3.9617	**75° 50′**
20′	.2476	.9689	.2555	3.9136	**40′**
30′	.2504	.9681	.2586	3.8667	**30′**
40′	.2532	.9674	.2617	3.8208	**20′**
50′	.2560	.9667	.2648	3.7760	**75° 10′**
15° 0′	.2588	.9659	.2679	3.7321	**75° 0′**
10′	.2616	.9652	.2711	3.6891	**74° 50′**
20′	.2644	.9644	.2742	3.6470	**40′**
30′	.2672	.9636	.2773	3.6059	**30′**
40′	.2700	.9628	.2805	3.5656	**20′**
50′	.2728	.9621	.2836	3.5261	**74° 10′**
16° 0′	.2756	.9613	.2867	3.4874	**74° 0′**
10′	.2784	.9605	.2899	3.4495	**73° 50′**
20′	.2812	.9596	.2931	3.4124	**40′**
30′	.2840	.9588	.2962	3.3759	**30′**
40′	.2868	.9580	.2994	3.3402	**20′**
50′	.2896	.9572	.3026	3.3052	**73° 10′**
17° 0′	.2924	.9563	.3057	3.2709	**73° 0′**
10′	.2952	.9555	.3089	3.2371	**72° 50′**
20′	.2979	.9546	.3121	3.2041	**40′**
30′	.3007	.9537	.3153	3.1716	**30′**
40′	.3035	.9528	.3185	3.1397	**20′**
50′	.3062	.9520	.3217	3.1084	**72° 10′**
18° 0′	.3090	.9511	.3249	3.0777	**72° 0′**
ANGLE	COS	SIN	COT	TAN	ANGLE

TABLE III (*Continued*)

Angle	Sin	Cos	Tan	Cot	Angle
18° 0′	.3090	.9511	.3249	3.0777	**72° 0′**
10′	.3118	.9502	.3281	3.0475	**71° 50′**
20′	.3145	.9492	.3314	3.0178	**40′**
30′	.3173	.9483	.3346	2.9887	**30′**
40′	.3201	.9474	.3378	2.9600	**20′**
50′	.3228	.9465	.3411	2.9319	**71° 10′**
19° 0′	.3256	.9455	.3443	2.9042	**71° 0′**
10′	.3283	.9446	.3476	2.8770	**70° 50′**
20′	.3311	.9436	.3508	2.8502	**40′**
30′	.3338	.9426	.3541	2.8239	**30′**
40′	.3365	.9417	.3574	2.7980	**20′**
50′	.3393	.9407	.3607	2.7725	**70° 10′**
20° 0′	.3420	.9397	.3640	2.7475	**70° 0′**
10′	.3448	.9387	.3673	2.7228	**69° 50′**
20′	.3475	.9377	.3706	2.6985	**40′**
30′	.3502	.9367	.3739	2.6746	**30′**
40′	.3529	.9356	.3772	2.6511	**20′**
50′	.3557	.9346	.3805	2.6279	**69° 10′**
21° 0′	.3584	.9336	.3839	2.6051	**69° 0′**
10′	.3611	.9325	.3872	2.5826	**68° 50′**
20′	.3638	.9315	.3906	2.5605	**40′**
30′	.3665	.9304	.3939	2.5386	**30′**
40′	.3692	.9293	.3973	2.5172	**20′**
50′	.3719	.9283	.4006	2.4960	**68° 10′**
22° 0′	.3746	.9272	.4040	2.4751	**68° 0′**
10′	.3773	.9261	.4074	2.4545	**67° 50′**
20′	.3800	.9250	.4108	2.4342	**40′**
30′	.3827	.9239	.4142	2.4142	**30′**
40′	.3854	.9228	.4176	2.3945	**20′**
50′	.3881	.9216	.4210	2.3750	**67° 10′**
23° 0′	.3907	.9205	.4245	2.3559	**67° 0′**
10′	.3934	.9194	.4279	2.3369	**66° 50′**
20′	.3961	.9182	.4314	2.3183	**40′**
30′	.3987	.9171	.4348	2.2998	**30′**
40′	.4014	.9159	.4383	2.2817	**20′**
50′	.4041	.9147	.4417	2.2637	**66° 10′**
24° 0′	.4067	.9135	.4452	2.2460	**66° 0′**
10′	.4094	.9124	.4487	2.2286	**65° 50′**
20′	.4120	.9112	.4522	2.2113	**40′**
30′	.4147	.9100	.4557	2.1943	**30′**
40′	.4173	.9088	.4592	2.1775	**20′**
50′	.4200	.9075	.4628	2.1609	**65° 10′**
25° 0′	.4226	.9063	.4663	2.1445	**65° 0′**
10′	.4253	.9051	.4699	2.1283	**64° 50′**
20′	.4279	.9038	.4734	2.1123	**40′**
30′	.4305	.9026	.4770	2.0965	**30′**
40′	.4331	.9013	.4806	2.0809	**20′**
50′	.4358	.9001	.4841	2.0655	**64° 10′**
26° 0′	.4384	.8988	.4877	2.0503	**64° 0′**
10′	.4410	.8975	.4913	2.0353	**63° 50′**
20′	.4436	.8962	.4950	2.0204	**40′**
30′	.4462	.8949	.4986	2.0057	**30′**
40′	.4488	.8936	.5022	1.9912	**20′**
50′	.4514	.8923	.5059	1.9768	**63° 10′**
27° 0′	.4540	.8910	.5095	1.9626	**63° 0′**
Angle	Cos	Sin	Cot	Tan	Angle

TABLE III (*Continued*)

Angle	Sin	Cos	Tan	Cot	Angle
27° 0′	.4540	.8910	.5095	1.9626	**63° 0′**
10′	.4566	.8897	.5132	1.9486	**62° 50′**
20′	.4592	.8884	.5169	1.9347	**40′**
30′	.4617	.8870	.5206	1.9210	**30′**
40′	.4643	.8857	.5243	1.9074	**20′**
50′	.4669	.8843	.5280	1.8940	**62° 10′**
28° 0′	.4695	.8829	.5317	1.8807	**62° 0′**
10′	.4720	.8816	.5354	1.8676	**61° 50′**
20′	.4746	.8802	.5392	1.8546	**40′**
30′	.4772	.8788	.5430	1.8418	**30′**
40′	.4797	.8774	.5467	1.8291	**20′**
50′	.4823	.8760	.5505	1.8165	**61° 10′**
29° 0′	.4848	.8746	.5543	1.8040	**61° 0′**
10′	.4874	.8732	.5581	1.7917	**60° 50′**
20′	.4899	.8718	.5619	1.7796	**40′**
30′	.4924	.8704	.5658	1.7675	**30′**
40′	.4950	.8689	.5696	1.7556	**20′**
50′	.4975	.8675	.5735	1.7437	**60° 10′**
30° 0′	.5000	.8660	.5774	1.7321	**60° 0′**
10′	.5025	.8646	.5812	1.7205	**59° 50′**
20′	.5050	.8631	.5851	1.7090	**40′**
30′	.5075	.8616	.5890	1.6977	**30′**
40′	.5100	.8601	.5930	1.6864	**20′**
50′	.5125	.8587	.5969	1.6753	**59° 10′**
31° 0′	.5150	.8572	.6009	1.6643	**59° 0′**
10′	.5175	.8557	.6048	1.6534	**58° 50′**
20′	.5200	.8542	.6088	1.6426	**40′**
30′	.5225	.8526	.6128	1.6319	**30′**
40′	.5250	.8511	.6168	1.6212	**20′**
50′	.5275	.8496	.6208	1.6107	**58° 10′**
32° 0′	.5299	.8480	.6249	1.6003	**58° 0′**
10′	.5324	.8465	.6289	1.5900	**57° 50′**
20′	.5348	.8450	.6330	1.5798	**40′**
30′	.5373	.8434	.6371	1.5697	**30′**
40′	.5398	.8418	.6412	1.5597	**20′**
50′	.5422	.8403	.6453	1.5497	**57° 10′**
33° 0′	.5446	.8387	.6494	1.5399	**57° 0′**
10′	.5471	.8371	.6536	1.5301	**56° 50′**
20′	.5495	.8355	.6577	1.5204	**40′**
30′	.5519	.8339	.6619	1.5108	**30′**
40′	.5544	.8323	.6661	1.5013	**20′**
50′	.5568	.8307	.6703	1.4919	**56° 10′**
34° 0′	.5592	.8290	.6745	1.4826	**56° 0′**
10′	.5616	.8274	.6787	1.4733	**55° 50′**
20′	.5640	.8258	.6830	1.4641	**40′**
30′	.5664	.8241	.6873	1.4550	**30′**
40′	.5688	.8225	.6916	1.4460	**20′**
50′	.5712	.8208	.6959	1.4370	**55° 10′**
35° 0′	.5736	.8192	.7002	1.4281	**55° 0′**
10′	.5760	.8175	.7046	1.4193	**54° 50′**
20′	.5783	.8158	.7089	1.4106	**40′**
30′	.5807	.8141	.7133	1.4019	**30′**
40′	.5831	.8124	.7177	1.3934	**20′**
50′	.5854	.8107	.7221	1.3848	**54° 10′**
36° 0′	.5878	.8090	.7265	1.3764	**54° 0′**
Angle	Cos	Sin	Cot	Tan	Angle

TABLE III (*Continued*)

Angle	Sin	Cos	Tan	Cot	Angle
36° 0′	.5878	.8090	.7265	1.3764	**54° 0′**
10′	.5901	.8073	.7310	1.3680	**53° 50′**
20′	.5925	.8056	.7355	1.3597	**40′**
30′	.5948	.8039	.7400	1.3514	**30′**
40′	.5972	.8021	.7445	1.3432	**20′**
50′	.5995	.8004	.7490	1.3351	**53° 10′**
37° 0′	.6018	.7986	.7536	1.3270	**53° 0′**
10′	.6041	.7969	.7581	1.3190	**52° 50′**
20′	.6065	.7951	.7627	1.3111	**40′**
30′	.6088	.7934	.7673	1.3032	**30′**
40′	.6111	.7916	.7720	1.2954	**20′**
50′	.6134	.7898	.7766	1.2876	**52° 10′**
38° 0′	.6157	.7880	.7813	1.2799	**52° 0′**
10′	.6180	.7862	.7860	1.2723	**51° 50′**
20′	.6202	.7844	.7907	1.2647	**40′**
30′	.6225	.7826	.7954	1.2572	**30′**
40′	.6248	.7808	.8002	1.2497	**20′**
50′	.6271	.7790	.8050	1.2423	**51° 10′**
39° 0′	.6293	.7771	.8098	1.2349	**51° 0′**
10′	.6316	.7753	.8146	1.2276	**50° 50′**
20′	.6338	.7735	.8195	1.2203	**40′**
30′	.6361	.7716	.8243	1.2131	**30′**
40′	.6383	.7698	.8292	1.2059	**20′**
50′	.6406	.7679	.8342	1.1988	**50° 10′**
40° 0′	.6428	.7660	.8391	1.1918	**50° 0′**
10′	.6450	.7642	.8441	1.1847	**49° 50′**
20′	.6472	.7623	.8491	1.1778	**40′**
30′	.6494	.7604	.8541	1.1708	**30′**
40′	.6517	.7585	.8591	1.1640	**20′**
50′	.6539	.7566	.8642	1.1571	**49° 10′**
41° 0′	.6561	.7547	.8693	1.1504	**49° 0′**
10′	.6583	.7528	.8744	1.1436	**48° 50′**
20′	.6604	.7509	.8796	1.1369	**40′**
30′	.6626	.7490	.8847	1.1303	**30′**
40′	.6648	.7470	.8899	1.1237	**20′**
50′	.6670	.7451	.8952	1.1171	**48° 10′**
42° 0′	.6691	.7431	.9004	1.1106	**48° 0′**
10′	.6713	.7412	.9057	1.1041	**47° 50′**
20′	.6734	.7392	.9110	1.0977	**40′**
30′	.6756	.7373	.9163	1.0913	**30′**
40′	.6777	.7353	.9217	1.0850	**20′**
50′	.6799	.7333	.9271	1.0786	**47° 10′**
43° 0′	.6820	.7314	.9325	1.0724	**47° 0′**
10′	.6841	.7294	.9380	1.0661	**46° 50′**
20′	.6862	.7274	.9435	1.0599	**40′**
30′	.6884	.7254	.9490	1.0538	**30′**
40′	.6905	.7234	.9545	1.0477	**20′**
50′	.6926	.7214	.9601	1.0416	**46° 10′**
44° 0′	.6947	.7193	.9657	1.0355	**46° 0′**
10′	.6967	.7173	.9713	1.0295	**45° 50′**
20′	.6988	.7153	.9770	1.0235	**40′**
30′	.7009	.7133	.9827	1.0176	**30′**
40′	.7030	.7112	.9884	1.0117	**20′**
50′	.7050	.7092	.9942	1.0058	**45° 10′**
45° 0′	.7071	.7071	1.0000	1.0000	**45° 0′**
Angle	Cos	Sin	Cot	Tan	Angle

TABLE IV. LOGARITHMS OF THE SINES, COSINES, TANGENTS, AND COTANGENTS OF CERTAIN ANGLES (INCREASED BY 10)

ANGLE	LOG SIN	LOG COS	LOG TAN	LOG COT	ANGLE
0° 0′					**90° 0′**
10′	7.4637	10.0000	7.4637	12.5363	**89° 50′**
20′	.7648	.0000	.7648	.2352	**40′**
30′	.9408	.0000	.9409	.0591	**30′**
40′	8.0658	.0000	8.0658	11.9342	**20′**
50′	.1627	.0000	.1627	.8373	**89° 10′**
1° 0′	8.2419	9.9999	8.2419	11.7581	**89° 0′**
10′	.3088	.9999	.3089	.6911	**88° 50′**
20′	.3668	.9999	.3669	.6331	**40′**
30′	.4179	.9999	.4181	.5819	**30′**
40′	.4637	.9998	.4638	.5362	**20′**
50′	.5050	.9998	.5053	.4947	**88° 10′**
2° 0′	8.5428	9.9997	8.5431	11.4569	**88° 0′**
10′	.5776	.9997	.5779	.4221	**87° 50′**
20′	.6097	.9996	.6101	.3899	**40′**
30′	.6397	.9996	.6401	.3599	**30′**
40′	.6677	.9995	.6682	.3318	**20′**
50′	.6940	.9995	.6945	.3055	**87° 10′**
3° 0′	8.7188	9.9994	8.7194	11.2806	**87° 0′**
10′	.7423	.9993	.7429	.2571	**86° 50′**
20′	.7645	.9993	.7652	.2348	**40′**
30′	.7857	.9992	.7865	.2135	**30′**
40′	.8059	.9991	.8067	.1933	**20′**
50′	.8251	.9990	.8261	.1739	**86° 10′**
4° 0′	8.8436	9.9989	8.8446	11.1554	**86° 0′**
10′	.8613	.9989	.8624	.1376	**85° 50′**
20′	.8783	.9988	.8795	.1205	**40′**
30′	.8946	.9987	.8960	.1040	**30′**
40′	.9104	.9986	.9118	.0882	**20′**
50′	.9256	.9985	.9272	.0728	**85° 10′**
5° 0′	8.9403	9.9983	8.9420	11.0580	**85° 0′**
10′	.9545	.9982	.9563	.0437	**84° 50′**
20′	.9682	.9981	.9701	.0299	**40′**
30′	.9816	.9980	.9836	.0164	**30′**
40′	.9945	.9979	.9966	.0034	**20′**
50′	9.0070	.9977	9.0093	10.9907	**84° 10′**
6° 0′	9.0192	9.9976	9.0216	.9784	**84° 0′**
10′	.0311	.9975	.0336	10.9664	**83° 50′**
20′	.0426	.9973	.0453	.9547	**40′**
30′	.0539	.9972	.0567	.9433	**30′**
40′	.0648	.9971	.0678	.9322	**20′**
50′	.0755	.9969	.0786	.9214	**83° 10′**
7° 0′	9.0859	9.9968	9.0891	10.9109	**83° 0′**
10′	.0961	.9966	.0995	.9005	**82° 50′**
20′	.1060	.9964	.1096	.8904	**40′**
30′	.1157	.9963	.1194	.8806	**30′**
40′	.1252	.9961	.1291	.8709	**20′**
50′	.1345	.9959	.1385	.8615	**82° 10′**
8° 0′	9.1436	9.9958	9.1478	10.8522	**82° 0′**
10′	.1525	.9956	.1569	.8431	**81° 50′**
20′	.1612	.9954	.1658	.8342	**40′**
30′	.1697	.9952	.1745	.8255	**30′**
40′	.1781	.9950	.1831	.8169	**20′**
50′	.1863	.9948	.1915	.8085	**81° 10′**
9° 0′	9.1943	9.9946	9.1997	10.8003	**81° 0′**
ANGLE	LOG COS	LOG SIN	LOG COT	LOG TAN	ANGLE

TABLE IV (*Continued*)

ANGLE	LOG SIN	LOG COS	LOG TAN	LOG COT	ANGLE
9° 0′	9.1943	9.9946	9.1997	10.8003	**81° 0′**
10′	.2022	.9944	.2078	.7922	**80° 50′**
20′	.2100	.9942	.2158	.7842	**40′**
30′	.2176	.9940	.2236	.7764	**30′**
40′	.2251	.9938	.2313	.7687	**20′**
50′	.2324	.9936	.2389	.7611	**80° 10′**
10° 0′	9.2397	9.9934	9.2463	10.7537	**80° 0′**
10′	.2468	.9931	.2536	.7464	**79° 50′**
20′	.2538	.9929	.2609	.7391	**40′**
30′	.2606	.9927	.2680	.7320	**30′**
40′	.2674	.9924	.2750	.7250	**20′**
50′	.2740	.9922	.2819	.7181	**79° 10′**
11° 0′	9.2806	9.9919	9.2887	10.7113	**79° 0′**
10′	.2870	.9917	.2953	.7047	**78° 50′**
20′	.2934	.9914	.3020	.6980	**40′**
30′	.2997	.9912	.3085	.6915	**30′**
40′	.3058	.9909	.3149	.6851	**20′**
50′	.3119	.9907	.3212	.6788	**78° 10′**
12° 0′	9.3179	9.9904	9.3275	10.6725	**78° 0′**
10′	.3238	.9901	.3336	.6664	**77° 50′**
20′	.3296	.9899	.3397	.6603	**40′**
30′	.3353	.9896	.3458	.6542	**30′**
40′	.3410	.9893	.3517	.6483	**20′**
50′	.3466	.9890	.3576	.6424	**77° 10′**
13° 0′	9.3521	9.9887	9.3634	10.6366	**77° 0′**
10′	.3575	.9884	.3691	.6309	**76° 50′**
20′	.3629	.9881	.3748	.6252	**40′**
30′	.3682	.9878	.3804	.6196	**30′**
40′	.3734	.9875	.3859	.6141	**20′**
50′	.3786	.9872	.3914	.6086	**76° 10′**
14° 0′	9.3837	9.9869	9.3968	10.6032	**76° 0′**
10′	.3887	.9866	.4021	.5979	**75° 50′**
20′	.3937	.9863	.4074	.5926	**40′**
30′	.3986	.9859	.4127	.5873	**30′**
40′	.4035	.9856	.4178	.5822	**20′**
50′	.4083	.9853	.4230	.5770	**75° 10′**
15° 0′	9.4130	9.9849	9.4281	10.5719	**75° 0′**
10′	.4177	.9846	.4331	.5669	**74° 50′**
20′	.4223	.9843	.4381	.5619	**40′**
30′	.4269	.9839	.4430	.5570	**30′**
40′	.4314	.9836	.4479	.5521	**20′**
50′	.4359	.9832	.4527	.5473	**74° 10′**
16° 0′	9.4403	9.9828	9.4575	10.5425	**74° 0′**
10′	.4447	.9825	.4622	.5378	**73° 50′**
20′	.4491	.9821	.4669	.5331	**40′**
30′	.4533	.9817	.4716	.5284	**30′**
40′	.4576	.9814	.4762	.5238	**20′**
50′	.4618	.9810	.4808	.5192	**73° 10′**
17° 0′	9.4659	9.9806	9.4853	10.5147	**73° 0′**
10′	.4700	.9802	.4898	.5102	**72° 50′**
20′	.4741	.9798	.4943	.5057	**40′**
30′	.4781	.9794	.4987	.5013	**30′**
40′	.4821	.9790	.5031	.4969	**20′**
50′	.4861	.9786	.5075	.4925	**72° 10′**
18° 0′	9.4900	9.9782	9.5118	10.4882	**72° 0′**
ANGLE	LOG COS	LOG SIN	LOG COT	LOG TAN	ANGLE

TABLE IV (*Continued*)

ANGLE	LOG SIN	LOG COS	LOG TAN	LOG COT	ANGLE
18° 0′	9.4900	9.9782	9.5118	10.4882	**72° 0′**
10′	.4939	.9778	.5161	.4839	**71° 50′**
20′	.4977	.9774	.5203	.4797	**40′**
30′	.5015	.9770	.5245	.4755	**30′**
40′	.5052	.9765	.5287	.4713	**20′**
50′	.5090	.9761	.5329	.4671	**71° 10′**
19° 0′	9.5126	9.9757	9.5370	10.4630	**71° 0′**
10′	.5163	.9752	.5411	.4589	**70° 50′**
20′	.5199	.9748	.5451	.4549	**40′**
30′	.5235	.9743	.5491	.4509	**30′**
40′	.5270	.9739	.5531	.4469	**20′**
50′	.5306	.9734	.5571	.4429	**70° 10′**
20° 0′	9.5341	9.9730	9.5611	10.4389	**70° 0′**
10′	.5375	.9725	.5650	.4350	**69° 50′**
20′	.5409	.9721	.5689	.4311	**40′**
30′	.5443	.9716	.5727	.4273	**30′**
40′	.5477	.9711	.5766	.4234	**20′**
50′	.5510	.9706	.5804	.4196	**69° 10′**
21° 0′	9.5543	9.9702	9.5842	10.4158	**69° 0′**
10′	.5576	.9697	.5879	.4121	**68° 50′**
20′	.5609	.9692	.5917	.4083	**40′**
30′	.5641	.9687	.5954	.4046	**30′**
40′	.5673	.9682	.5991	.4009	**20′**
50′	.5704	.9677	.6028	.3972	**68° 10′**
22° 0′	9.5736	9.9672	9.6064	10.3936	**68° 0′**
10′	.5767	.9667	.6100	.3900	**67° 50′**
20′	.5798	.9661	.6136	.3864	**40′**
30′	.5828	.9656	.6172	.3828	**30′**
40′	.5859	.9651	.6208	.3792	**20′**
50′	.5889	.9646	.6243	.3757	**67° 10′**
23° 0′	9.5919	9.9640	9.6279	10.3721	**67° 0′**
10′	.5948	.9635	.6314	.3686	**66° 50′**
20′	.5978	.9629	.6348	.3652	**40′**
30′	.6007	.9624	.6383	.3617	**30′**
40′	.6036	.9618	.6417	.3583	**20′**
50′	.6065	.9613	.6452	.3548	**66° 10′**
24° 0′	9.6093	9.9607	9.6486	10.3514	**66° 0′**
10′	.6121	.9602	.6520	.3480	**65° 50′**
20′	.6149	.9596	.6553	.3447	**40′**
30′	.6177	.9590	.6587	.3413	**30′**
40′	.6205	.9584	.6620	.3380	**20′**
50′	.6232	.9579	.6654	.3346	**65° 10′**
25° 0′	9.6259	9.9573	9.6687	10.3313	**65° 0′**
10′	.6286	.9567	.6720	.3280	**64° 50′**
20′	.6313	.9561	.6752	.3248	**40′**
30′	.6340	.9555	.6785	.3215	**30′**
40′	.6366	.9549	.6817	.3183	**20′**
50′	.6392	.9543	.6850	.3150	**64° 10′**
26° 0′	9.6418	9.9537	9.6882	10.3118	**64° 0′**
10′	.6444	.9530	.6914	.3086	**63° 50′**
20′	.6470	.9524	.6946	.3054	**40′**
30′	.6495	.9518	.6977	.3023	**30′**
40′	.6521	.9512	.7009	.2991	**20′**
50′	.6546	.9505	.7040	.2960	**63° 10′**
27° 0′	9.6570	9.9499	9.7072	10.2928	**63° 0′**
ANGLE	LOG COS	LOG SIN	LOG COT	LOG TAN	ANGLE

TABLE IV (*Continued*)

Angle	Log sin	Log cos	Log tan	Log cot	Angle
27° 0′	9.6570	9.9499	9.7072	10.2928	**63° 0′**
10′	.6595	.9492	.7103	.2897	**62° 50′**
20′	.6620	.9486	.7134	.2866	**40′**
30′	.6644	.9479	.7165	.2835	**30′**
40′	.6668	.9473	.7196	.2804	**20′**
50′	.6692	.9466	.7226	.2774	**62° 10′**
28° 0′	9.6716	9.9459	9.7257	10.2743	**62° 0′**
10′	.6740	.9453	.7287	.2713	**61° 50′**
20′	.6763	.9446	.7317	.2683	**40′**
30′	.6787	.9439	.7348	.2652	**30′**
40′	.6810	.9432	.7378	.2622	**20′**
50′	.6833	.9425	.7408	.2592	**61° 10′**
29° 0′	9.6856	9.9418	9.7438	10.2562	**61° 0′**
10′	.6878	.9411	.7467	.2533	**60° 50′**
20′	.6901	.9404	.7497	.2503	**40′**
30′	.6923	.9397	.7526	.2474	**30′**
40′	.6946	.9390	.7556	.2444	**20′**
50′	.6968	.9383	.7585	.2415	**60° 10′**
30° 0′	9.6990	9.9375	9.7614	10.2386	**60° 0′**
10′	.7012	.9368	.7644	.2356	**59° 50′**
20′	.7033	.9361	.7673	.2327	**40′**
30′	.7055	.9353	.7701	.2299	**30′**
40′	.7076	.9346	.7730	.2270	**20′**
50′	.7097	.9338	.7759	.2241	**59° 10′**
31° 0′	9.7118	9.9331	9.7788	10.2212	**59° 0′**
10′	.7139	.9323	.7816	.2184	**58° 50′**
20′	.7160	.9315	.7845	.2155	**40′**
30′	.7181	.9308	.7873	.2127	**30′**
40′	.7201	.9300	.7902	.2098	**20′**
50′	.7222	.9292	.7930	.2070	**58° 10′**
32° 0′	9.7242	9.9284	9.7958	10.2042	**58° 0′**
10′	.7262	.9276	.7986	.2014	**57° 50′**
20′	.7282	.9268	.8014	.1986	**40′**
30′	.7302	.9260	.8042	.1958	**30′**
40′	.7322	.9252	.8070	.1930	**20′**
50′	.7342	.9244	.8097	.1903	**57° 10′**
33° 0′	9.7361	9.9236	9.8125	10.1875	**57° 0′**
10′	.7380	.9228	.8153	.1847	**56° 50′**
20′	.7400	.9219	.8180	.1820	**40′**
30′	.7419	.9211	.8208	.1792	**30′**
40′	.7438	.9203	.8235	.1765	**20′**
50′	.7457	.9194	.8263	.1737	**56° 10′**
34° 0′	9.7476	9.9186	9.8290	10.1710	**56° 0′**
10′	.7494	.9177	.8317	.1683	**55° 50′**
20′	.7513	.9169	.8344	.1656	**40′**
30′	.7531	.9160	.8371	.1629	**30′**
40′	.7550	.9151	.8398	.1602	**20′**
50′	.7568	.9142	.8425	.1575	**55° 10′**
35° 0′	9.7586	9.9134	9.8452	10.1548	**55° 0′**
10′	.7604	.9125	.8479	.1521	**54° 50′**
20′	.7622	.9116	.8506	.1494	**40′**
30′	.7640	.9107	.8533	.1467	**30′**
40′	.7657	.9098	.8559	.1441	**20′**
50′	.7675	.9089	.8586	.1414	**54° 10′**
36° 0′	9.7692	9.9080	9.8613	10.1387	**54° 0′**
Angle	Log cos	Log sin	Log cot	Log tan	Angle

TABLE IV (*Continued*)

ANGLE	LOG SIN	LOG COS	LOG TAN	LOG COT	ANGLE
36° 0′	9.7692	9.9080	9.8613	10.1387	**54° 0′**
10′	.7710	.9070	.8639	.1361	**53° 50′**
20′	.7727	.9061	.8666	.1334	**40′**
30′	.7744	.9052	.8692	.1308	**30′**
40′	.7761	.9042	.8718	.1282	**20′**
50′	.7778	.9033	.8745	.1255	**53° 10′**
37° 0′	9.7795	9.9023	9.8771	10.1229	**53° 0′**
10′	.7811	.9014	.8797	.1203	**52° 50′**
20′	.7828	.9004	.8824	.1176	**40′**
30′	.7844	.8995	.8850	.1150	**30′**
40′	.7861	.8985	.8876	.1124	**20′**
50′	.7877	.8975	.8902	.1098	**52° 10′**
38° 0′	9.7893	9.8965	9.8928	10.1072	**52° 0′**
10′	.7910	.8955	.8954	.1046	**51° 50′**
20′	.7926	.8945	.8980	.1020	**40′**
30′	.7941	.8935	.9006	.0994	**30′**
40′	.7957	.8925	.9032	.0968	**20′**
50′	.7973	.8915	.9058	.0942	**51° 10′**
39° 0′	9.7989	9.8905	9.9084	10.0916	**51° 0′**
10′	.8004	.8895	.9110	.0890	**50° 50′**
20′	.8020	.8884	.9135	.0865	**40′**
30′	.8035	.8874	.9161	.0839	**30′**
40′	.8050	.8864	.9187	.0813	**20′**
50′	.8066	.8853	.9212	.0788	**50° 10′**
40° 0′	9.8081	9.8843	9.9238	10.0762	**50° 0′**
10′	.8096	.8832	.9264	.0736	**49° 50′**
20′	.8111	.8821	.9289	.0711	**40′**
30′	.8125	.8810	.9315	.0685	**30′**
40′	.8140	.8800	.9341	.0659	**20′**
50′	.8155	.8789	.9366	.0634	**49° 10′**
41° 0′	9.8169	9.8778	9.9392	10.0608	**49° 0′**
10′	.8184	.8767	.9417	.0583	**48° 50′**
20′	.8198	.8756	.9443	.0557	**40′**
30′	.8213	.8745	.9468	.0532	**30′**
40′	.8227	.8733	.9494	.0506	**20′**
50′	.8241	.8722	.9519	.0481	**48° 10′**
42° 0′	9.8255	9.8711	9.9544	10.0456	**48° 0′**
10′	.8269	.8699	.9570	.0430	**47° 50′**
20′	.8283	.8688	.9595	.0405	**40′**
30′	.8297	.8676	.9621	.0379	**30′**
40′	.8311	.8665	.9646	.0354	**20′**
50′	.8324	.8653	.9671	.0329	**47° 10′**
43° 0′	9.8338	9.8641	9.9697	10.0303	**47° 0′**
10′	.8351	.8629	.9722	.0278	**46° 50′**
20′	.8365	.8618	.9747	.0253	**40′**
30′	.8378	.8606	.9773	.0228	**30′**
40′	.8391	.8594	.9798	.0202	**20′**
50′	.8405	.8582	.9823	.0177	**46° 10′**
44° 0′	9.8418	9.8569	9.9848	10.0152	**46° 0′**
10′	.8431	.8557	.9874	.0126	**45° 50′**
20′	.8444	.8545	.9899	.0101	**40′**
30′	.8457	.8532	.9924	.0076	**30′**
40′	.8469	.8520	.9949	.0051	**20′**
50′	.8482	.8507	.9975	.0025	**45° 10′**
45° 0′	9.8495	9.8495	10.0000	10.0000	**45° 0′**
ANGLE	LOG COS	LOG SIN	LOG COT	LOG TAN	ANGLE

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